1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega$ of class $C^1$. Show that a function $u$ belongs to $W^{1,\infty}(\Omega)$ iff $u$ is Lipschitz continuous on $\Omega$.

2. Give an example of a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous on $\Omega$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega$ of class $C^2$. Show that the boundary value problem for the biharmonic equation

$$\Delta \Delta u = f \text{ in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

with $f \in L^2(\Omega)$ has a unique weak solution $u \in W^{2,2}_0(\Omega)$, where weak solution means that

$$\int_{\Omega} \Delta u \Delta v = \int_{\Omega} fv,$$

for all $v \in C_c^\infty(\Omega)$.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that a function $u \in L^2_{\text{loc}}(\Omega)$ satisfies

$$\int_{\Omega} u \Delta v = \int_{\Omega} fv,$$

for all $v \in C_c^\infty(\Omega)$, where $f \in C^\infty(\Omega)$. Prove that $u \in C^\infty(\Omega)$ and $\Delta u = f$ holds on $\Omega$.

5. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ which satisfies weakly

$$D_i(a^{ij}(x)D_j u) = 0,$$

with $(a^{ij})$ bounded and uniformly elliptic. Prove that if $\int_{\mathbb{R}^n} u^2 < \infty$ then $u$ is a constant.