1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C^\infty(\Omega)$ with $u = 0$ on $\partial \Omega$. Prove that for every $1 \leq p < \frac{n}{n-1}$ there is a constant $C(n,p)$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|\Delta u\|_{L^1(\Omega)}.$$ 

You can think of this result as a weak replacement for the elliptic $L^1$ estimate for $\Delta$, which is false, and which would bound the $W^{2,1}$ norm of $u$ (and therefore also the $W^{1,\frac{n}{n-1}}$ norm by Sobolev) by the $L^1$ norm of $\Delta u$.

2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and $u \in W^{2,2}_{\text{loc}}(\Omega)$ be such that $\Delta u \in W^{1,n}_{\text{loc}}(\Omega)$. Show that $D^2 u \in L^{2,n}_{\text{loc}}(\Omega)$.

3. Show that the function $\log |x|, x \in \mathbb{R}$ lies in $L^{1,1}(\mathbb{R})$.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in W^{1,2}(\Omega)$ a weak solution of

$$D_i(a^{ij}D_j u) = g + D_i f^i,$$

where $a^{ij}$ are constant and elliptic, $f^i \in L^{2,\lambda+2}(\Omega)$ and $g \in L^{2,\lambda}(\Omega)$, with $0 \leq \lambda < n$. Show that $Du \in L^{2,\lambda+2}_{\text{loc}}(\Omega)$, and furthermore for any relatively compact domain $\Omega' \subset \Omega$ we have

$$\|Du\|_{L^{2,\lambda+2}(\Omega')} \leq C(\|u\|_{L^2(\Omega)} + \|g\|_{L^{2,\lambda}(\Omega)} + \|f\|_{L^{2,\lambda+2}(\Omega)}),$$

where $C$ depends only on $a^{ij}, d(\Omega', \partial \Omega), n, \lambda$. 
