TRUTH ASSIGNMENTS AND LOGICAL IMPLICATION

Let us consider the question of semantics; the interpretation of the sentences in our model language in terms of their truth or falsity in some possible world.

1. WHAT IS TRUTH?

DEFINITION: If \( S \) is a set of sentence symbols, then a truth assignment for \( S \) is a function \( \ddot{n} : S \rightarrow \{T, F\} \).

THEOREM: For every truth assignment \( \ddot{n} \) for \( S \), there exists a unique extension of \( \ddot{n} \) from \( S \) to \( \ddot{S} \), \( \ddot{n} : \ddot{S} \rightarrow \{T, F\} \), such that

1. \( \ddot{n}(\neg a) = T \) if \( \ddot{n}(a) = F \).
2. \( \ddot{n}(a \lor b) = T \) if \( \ddot{n}(a) = T \) and \( \ddot{n}(b) = T \).
3. \( \ddot{n}(a \land b) = T \) if \( \ddot{n}(a) = T \) or \( \ddot{n}(b) = T \).
4. \( \ddot{n}(a \rightarrow b) = F \) if \( \ddot{n}(a) = T \) and \( \ddot{n}(b) = F \).
5. \( \ddot{n}(a \leftrightarrow b) = T \) if \( \ddot{n}(a) = \ddot{n}(b) \).

Let us summarize:

1. We begin with an alphabet \( A \), consisting of logical symbols \( L \) (the sentential connectives and the parentheses), and atomic sentences \( S \).

2. From these we generate the expressions \( E \) all possible finite sequences of symbols in the alphabet.

3. Employing the sentential connectives we define five formula-building operations corresponding to each of the five connectives: \( E_1 \) (a unary operation) and \( E_2, E_3, E_4, E_5 \) (binary operations).
4. A set of expressions $I$ is said to be inductive if it contains the atomic sentences $S$ and is closed under the application of the formula-building operations; i.e., if $[]$, $[.]$ $I$, then $E_1([[]]) \subseteq I$, and $E_i([.] , [.] ) \subseteq I$, $i = 2, 3, 4, 5$.

The set of sentences $\bar{S}$ is then defined to be the intersection of all inductive sets.

5. A truth assignment is a function $[] : S \rightarrow \{T, F\}$ from the atomic sentences to the set of truth values. It can be shown that there exists a unique extension of $[]$ to a function $[] : \bar{S} \rightarrow \{T, F\}$ which is consistent with the sentential connectives.

2 Logical Implication

So far, so good. Now that we have established a framework, we can begin to discuss the idea of implication or proof. The basic idea is that we will introduce two such notions, apparently very different but which actually coincide.

**Definition:** A truth assignment $[]$ satisfies $[]$ if $[]([]) = T$. A sentence is a tautology if every truth assignment satisfies it. A set $[]$ of sentences tautologically implies another sentence $[]$ if and only if every truth assignment which satisfies every member of $[]$ also satisfies $[]$; in this case we write $[] \models []$. If $[]$ tautologically implies $[]$ and $[]$ tautologically implies $[]$, then $[]$ and $[]$ are said to be tautologically equivalent, denoted $[] \models =[]$.

Tautological equivalence is an example of a so-called "equivalence relation": that is, it satisfies the following three properties (where $[]$, $[.]$, $[.] [.] S$):

1. $[] \models =[]$.
2. If $[] \models =[]$, then $[] \models =[]$.
3. If $[] \models =[]$ and $[] \models =[]$ then $[] \models =[]$.

Recall that if $A$ and $B$ are sets, a relation (between elements of $A$ and elements of $B$) is any subset $R \subseteq A \times B$. For example, if $A = B = \mathbb{N}$ is the set of natural numbers, then "a divides b" states a relationship that holds for certain pairs of integers $(a, b)$, and does not hold for others; the set $R = \{<a, b> : a \mid b\}$ consists of all ordered pairs of integers $<a, b>$ which satisfy this relation, and may be thought of
as the set-theoretic equivalent of the verbal specification of the relation. A relation $R \subseteq A \times A$ is said to be a **binary relation** on $A$ (as opposed to an "n-ary" relation on $A$, which is a subset of the set of ordered $n$-tuples $A^{(n)}$).

A binary relation $R$ on $A$ is said to be an **equivalence relation** if it is:

1. **REFLEXIVE**: $\forall x \in A, <x, x> \in R$.
2. **SYMMETRIC**: $\forall x, y \in A$, if $<x, y> \in R$, then $<y, x> \in R$.
3. **TRANSITIVE**: $\forall x, y, z \in A$, if $<x, y> \in R$ and $<y, z> \in R$, then $<x, z> \in R$.

If $R$ is an equivalence relation on $A$, a common alternative notation is to write $x \sim y$ iff $<x, y> \in R$.

**Example**: If $n$ is a nonzero integer, two integers $x$ and $y$ are said to be congruent mod $n$ (written $x \equiv y \mod n$) iff $n \mid (x - y)$.

**EXERCISE**: Check that this defines an equivalence relation on $\mathbb{Z}$.

Specifying an equivalence relation on a set is the same thing as partitioning the set into disjoint pieces. Namely, for each $x \in A$, consider the set $A_x = \{y \in A : x \sim y\}$. Then given $x, y \in A$, either

1. $A_x = A_y$, which is the case iff $x \sim y$; or
2. $A_x \cap A_y = \emptyset$, otherwise.

**EXAMPLE**: In congruence mod $n$ there are $n$ equivalence classes:

$$A_k =: \{nx + k : k \in \mathbb{Z}, 0 \leq k < n\}.$$

Thus if $n = 3$, the equivalence classes are

- $\{0, 3, 6, 9, 12, ...\}$
- $\{1, 4, 7, 10, 13, ...\}$
- $\{2, 5, 8, 11, 14, ...\}$. 
Conversely, given any partition of a set $A$ into disjoint subsets, one can define an equivalence relation by agreeing to say that two elements of $A$ are "equivalent" iff both lie in the same subset.

Thus the sentences are partitioned into equivalence classes under the relation of tautological equivalence; these equivalence classes can be given the structure of a Boolean algebra, where the various axioms correspond to provable tautologies.

Thus let $[\square]$ denote the equivalence classes of formulas equivalent to $\square$. We can define a negation operation $\neg[\square]$ on the set of equivalence classes by setting: $\neg[\square] = [\neg \square]$. That is, the negation of the equivalence class $[\square]$ is the equivalence class of $\neg \square$. In order for this equivalence class to be well-defined, we must show it is independent of the choice of representative $\square$ for the class $[\square]$. Namely, suppose $\models \square \models \square_1$. Then $\neg \square \models \models \neg \square_1$, hence $[\neg \square] = [\neg \square_1]$. Similarly, one can extend the operations of $\land$ and $\lor$ from formulas to their equivalence classes. Thus if $\square_1 \models \square_2$ and $\square_1 \models \square_2$, then $\square_1 \square_1 \models \square_2 \square_2$, and thus $[\square_1 \square_1] = [\square_2 \square_2]$; and similarly for disjunction $\lor$.

These three extended operations make the set of equivalence classes into a Boolean algebra. Verifying that the axioms for a Boolean algebra are satisfied reduces to verifying that certain pairs of formulas are tautological equivalences. For example, De Morgan's laws (named after the 19th century logician) assert that

$$\neg(\square \land \square) = (\neg \square) \lor (\neg \square).$$

1.7 Sentential Connectives

**DEFINITION**: A k-place Boolean function is a function $f : \{T, F\}^k \rightarrow \{T, F\}$.

Every sentence $\square$ can be identified with some Boolean function $f_\square$, and two sentences $\square$ and $\square$ are tautologically equivalent iff their corresponding Boolean functions coincide. The disjunctive normal form is a systematic way of expressing a wff so as to exhibit this correspondence. But it need not be the simplest.

Suppose ABCD is a short-hand for the two-place Boolean function which assigns $f(T, T) = A$, $f(T, F) = B$, $f(F, T) = C$, $f(F, F) = D$; let us express such Boolean functions in terms of a sentence constructed from the two simpler sentences $\square$ and $\square$: 
**DEFINITION:** A set of sentential connectives is complete if every Boolean function can be realized by a formula generated by the connectives in that set.

**THEOREM:** The sets \{\neg, \land\} and \{\neg, \lor\} are complete; \{\land, \lor\} is not complete.

The Sheffer stroke is a single sentential connective, which is complete standing by itself.

Given a set \{\} of wffs, there may or may not be a truth assignment which satisfies every member of that set (for example, if there is a wff \(\alpha\) such that both \(\alpha\) \(\land\) \(\alpha\) and \(\alpha\) \(\lor\) \(\neg\alpha\)). So we need to know when such an assignment can exist. This is the next topic to be considered.

**THE COMPACTNESS THEOREM:** If for every finite subset \(\mathcal{S}_0\) of \(\mathcal{S}\), there exists a truth assignment \(\tau\) which satisfies every member of \(\mathcal{S}_0\), then there exists a truth assignment which satisfies every member of \(\mathcal{S}\).