THE COMPACTNESS THEOREM

The following result is of fundamental importance because it enables us to pass from (potentially) infinite to finite sets of sentences.

THE COMPACTNESS THEOREM: A set of sentences \( S \) is satisfiable if and only if every finite subset of it is satisfiable.

Let us say that a set \( S \) of sentences is **finitely satisfiable** if every finite subset of it is satisfiable.

**LEMMA 1.** If \( S \) is finitely satisfiable, then either \( S ; a \) or \( S ; \neg a \) is finitely satisfiable.

**Proof:** If \( S ; a \) is not finitely satisfiable, then there exists a finite subset of \( S ; a \) which is not satisfied by any truth assignment \( n \). Now since every finite subset of \( S ; a \) is either of the form \( S_0 \) or \( S_0 ; a \), where \( S_0 \) is a finite subset of \( S \), and since every finite subset \( S_0 \) of \( S \) is satisfiable (because \( S \) is finitely satisfiable), it follows that there exists a finite subset \( S_0 \) of \( S \) such that \( S_0 ; a \) is not satisfiable. But since \( S_0 \), being finite, is satisfiable, it follows that every truth assignment \( n \) which satisfies \( S_0 \) does not satisfy \( a \). Thus:

If \( S ; a \) is not finitely satisfiable, there exists a finite subset \( S_0 \) of \( S \) such that every truth assignment \( n \) that satisfies \( S_0 \) does not satisfy \( a \).

Similarly, if \( S : \neg a \) is not finitely satisfiable, there exists a finite subset \( S_1 \) of \( S \) that every truth assignment \( n \) which satisfies \( S_1 \) does not satisfy \( \neg a \).

Now suppose that neither \( S ; a \) nor \( S ; \neg a \) is finitely satisfiable, let \( S_0 \) and \( S_1 \) be the sets indicated above that exist in each case, and consider the finite subset \( S_0 \).
Since \( \mathfrak{P} \) is finitely satisfiable, there exists a truth assignment \( \mathfrak{M} \) which satisfies \( \mathfrak{P}_0 \cup \mathfrak{P}_1 \); and \( \mathfrak{M} \) must also satisfy either \( \mathfrak{P} \) or \( \neg \mathfrak{P} \). Suppose \( \mathfrak{P} \) satisfies \( \mathfrak{P} \).

Then \( \mathfrak{P} \) satisfies \( \mathfrak{P}_0 \cup \mathfrak{P}_1 \); hence \( \mathfrak{P}_0 \) ; \( \mathfrak{P} \), which is impossible. This proves Lemma 1.

Let \( \mathfrak{P}_1, \mathfrak{P}_2, \ldots \) be an enumeration of the (countable) set of sentences. Use recursion to define \( \mathfrak{P}_0 =: \mathfrak{P} \), and

\[
\mathfrak{P}_{n+1} =: \mathfrak{P}_n: \mathfrak{P}_{n+1} \quad \text{if} \quad \mathfrak{P}_n: \mathfrak{P}_{n+1} \text{ satisfiable},
\]

\[
\mathfrak{P}_{n+1} =: \mathfrak{P}_n: \neg \mathfrak{P}_{n+1} \quad \text{if} \quad \mathfrak{P}_n: \mathfrak{P}_{n+1} \text{ not satisfiable}.
\]

It then follows from Lemma 1 that \( \mathfrak{P}_{n+1} \) satisfiable for all \( n \geq 0 \).

Let \( \mathfrak{P} =: \mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \ldots \). It is simple to see that the set \( \mathfrak{P} \) is finitely satisfiable: if \( \mathfrak{P}_0 \) is a finite subset of \( \mathfrak{P} \), then \( \mathfrak{P}_0 \cup \mathfrak{P}_n \) for some \( n \geq 0 \); and since \( \mathfrak{P}_n \) is finitely satisfiable, it follows that \( \mathfrak{P}_0 \), being a finite subset of it, is satisfiable. Furthermore, by construction the set \( \mathfrak{P} \) contains both (1) our initial set \( \mathfrak{P} \) and (2) for any other sentence \( \mathfrak{P} \), either the sentence or its negation; that is, for all \( \mathfrak{P} \), either \( \mathfrak{P} \) or \( \neg \mathfrak{P} \).

Note that the set \( \mathfrak{P} \) has been made as large as possible while remaining finitely satisfiable. This is because if \( \mathfrak{P} \) \( \mathfrak{P} \) then \( \neg \mathfrak{P} \) \( \mathfrak{P} \); thus one cannot also add \( \mathfrak{P} \) to \( \mathfrak{P} \) and yet have a finitely satisfiable set: no truth assignment \( \mathfrak{M} \) can simultaneously satisfy \( \mathfrak{P} \) and \( \neg \mathfrak{P} \). This key step is the clever idea behind the proof of the compactness theorem: one takes the original finitely satisfiable set \( \mathfrak{P} \) and expands it to a maximal finitely satisfiable set \( \mathfrak{P} \).

Next define the following truth assignment \( \mathfrak{M} \) on sentence symbols: Let \( \mathfrak{M}(A) = T \) if \( A \) \( \mathfrak{P} \), \( \mathfrak{M}(A) = F \) if \( A \) \( \neg \mathfrak{P} \). Suppose one can establish that for all sentences \( \mathfrak{P} \) \( \mathfrak{M} \) satisfies \( \mathfrak{P} \) \( \mathfrak{P} \) \( \mathfrak{P} \). Then \( \mathfrak{M} \) satisfies \( \mathfrak{P} \) and the compactness theorem is proved. That \( \mathfrak{M} \) satisfies this property is the content of Lemma 2.
Lemma 2. Let \( \Box \) be a finitely satisfiable set of wffs having the property that for every wff \( \Box \), either \( \Box \Box \Box \) or \( \neg \Box \Box \Box \) (but not both). Suppose that \( \Box \) is the truth assignment on the sentence symbols \( A \) taking the values \( \Box(A) = T \), \( A \Box \Box \) and \( \Box(A) = F \), \( A \Box \Box \). Then \( \Box \) has the property that \( \Box(\Box) = T \) if and only if \( \Box \Box \Box \).

Proof: Let \( B \) be the set of sentence symbols, let \( C \) be the set of sentences generated from \( B \) by the connectives, and define \( S =: \{ \Box : \Box \Box \Box \Box \Box = T \} \). The proof is by induction on the set \( S \). It is clear that \( B \subseteq S \), and that \( S \) is closed under negation (because reverses both membership in \( \Box \) and truth value); it therefore suffices to prove that \( S \) is closed under the binary connectives.

Let \( g : C \times C \rightarrow C \) denote one of the binary connective operations on sentences, and \( G : \{ T, F \} \times \{ T, F \} \rightarrow \{ T, F \} \) its corresponding two-place Boolean function. Suppose \( \Box_1, \Box_2 \in S \), let \( \Box_3 := g(\Box_1, \Box_2) \), and let

\[
\Box_i = \Box_{i-1}, \text{ if } \Box_{i-1} \Box_{i-1}, \text{ and } \Box = \neg \Box_1, \text{ if } \Box_{i-1} \Box_{i-1}, \quad \text{ for } i = 1, 2, 3.
\]

Note that the triplet set \( \{ \Box_1, \Box_2, \Box_3 \} \) is a finite subset of \( \Box \), hence satisfiable; let \( \Box \) be a truth assignment such that \( \Box(\Box) = T \), \( i = 1, 2, 3 \).

Step 1: Note that \( \Box(\Box_i) = \Box(\Box) \) for \( i = 1, 2, 3 \). For, first, if \( \Box_{i-1} \Box_{i-1} \), \( i = 1, 2 \), then \( \Box(\Box_i) = T \) (because it is assumed that \( \Box \Box \Box \subseteq S \), and \( \Box = \Box \), hence \( \Box(\Box) = \Box(\Box) = T \); while if \( \Box_{i-1} \Box_{i-1} \), then \( \Box(\Box_i) = F \), and \( \Box = \Box \), hence \( \Box(\Box) = \Box(\Box) = T \) and \( \Box(\Box) = F \). Finally, to see that \( \Box(g(\Box_1, \Box_2)) = \Box(\Box_3) \), note that

\[
\Box(\Box_3) = \Box(g(\Box_1, \Box_2)) = \Box(\Box(\Box_1), \Box(\Box_2)) = \Box(\Box(\Box_1), \Box(\Box_2)) = \Box(\Box(\Box_1), \Box(\Box_2)) = \Box(\Box(\Box_1), \Box(\Box_2)) = \Box(\Box_3).
\]

Step 2: Note that \( \Box_3 \Box \Box \Box \Box \Box \subseteq S \): if \( \Box_3 \Box \Box \Box \Box \Box \), then \( \Box_3 = \Box_3 \), hence \( \Box(\Box_3) = \Box(\Box) = \Box(\Box) = T \); and if \( \Box_3 \Box \Box \Box \Box \Box \), then \( \Box(\Box) = \Box_3 \), hence \( \Box(\Box) = \Box(\Box) = \Box(\Box) = T \), hence \( \Box(\Box) = F \).

This concludes the proof of Lemma 2.

Corollary: If \( \Box \models \Box \) then there exists a finite subset \( \Box_0 \Box \Box \Box \Box \Box \) such that \( \Box_0 \models \Box \Box \Box \Box \Box \Box \).
PROOF. Use the basic fact (an immediate consequence of the definitions):

\[ \emptyset \models \phi \text{ if and only if } \phi ; \neg \phi \text{ is not satisfiable} \]

Suppose then that \( \emptyset \models \phi \) and hence that \( \phi ; \neg \phi \) is not satisfiable. Then, by the compactness theorem, there exists a finite subset of \( \emptyset ; \neg \emptyset \) that is not satisfiable and is of the form \( \emptyset_0 ; \neg \emptyset \) for some finite subset \( \emptyset_0 \) of \( \emptyset \). (By the compactness theorem, there exists a finite subset \( \emptyset^* \) of \( \emptyset ; \neg \emptyset \) that is not satisfiable. Let \( \emptyset_0 =: \emptyset^* \). The set \( \emptyset_0 \) is of course finite; if \( \emptyset_0 = \emptyset^* \), that is, if the unsatisfiable set \( \emptyset^* \) does not contain \( \neg \emptyset \) then \( \neg \emptyset \) can be added to it, and the resulting set is still unsatisfiable.) But, once again by the basic fact, if \( \emptyset_0 ; \neg \emptyset \) is not satisfiable, then (the finite subset) \( \emptyset_0 \models \phi \).

Exercise 3. Prove the compactness theorem is a consequence of the corollary to the compactness theorem (and therefore is equivalent to it).

Solution: Suppose the set \( \emptyset \) is finitely satisfiable but not satisfiable. Let \( \emptyset \) be an arbitrary sentence. Then \( \emptyset \models \emptyset \models \phi \rightarrow \psi \), hence, by the corollary, there exists a finite subset \( \emptyset_0 \) of \( \emptyset_1 \) such that \( \emptyset_0 \models \emptyset \models \phi \models \neg \psi \). But, because \( \emptyset \) is finitely satisfiable, \( \emptyset_0 \) is satisfiable, hence exists a truth assignment \( \emptyset \) that satisfies \( \emptyset_0 \). But since \( \emptyset_0 \models \emptyset \models \phi \models \neg \psi \), the truth assignment \( \emptyset \) must also satisfy \( \emptyset \models \neg \phi \), which is impossible.

Remark. The compactness theorem is equivalent to an assertion that the set of truth assignments is compact in an appropriate topology. In order to see this, two restatements are necessary.

First, one says a topological space \( X \) is compact if every open cover of \( X \) has a finite subcover. Equivalently, this is the case whenever a collection of closed subsets has an empty intersection one can find a finite subcollection that also has an empty intersection. (This is known as the finite intersection property.)

Second, the contrapositive statement of the compactness theorem is that if a set of sentences \( \emptyset \) is not satisfiable, then there exists a finite subset \( \emptyset_0 \) of \( \emptyset \) that is also not satisfiable.

Let \( C \) be the set of sentences, and \( \emptyset \) a truth assignment on the set \( C \). Then the function \( \emptyset : C \rightarrow \{T, F\} \) can be thought of as a generalized sequence \( \{\emptyset(c) : c \in C\} \)
C}, or a point in the product space \(\{T, F\}^C\). If \(\{T, F\}\) is given the discrete topology, then it becomes a compact Hausdorff space; thus in turn, if the product space \(\{T, F\}^C\) is given the product topology, then by Tychonoff’s theorem the resulting topological space is also compact Hausdorff. Let the set of truth assignments \(\square \{T, F\}^C\) be given the induced topology; then it can be shown that \(\square\) is a closed subset of \(\{T, F\}^C\); thus it is itself compact.

If \(\Downarrow \in C\), let \(S_{\Downarrow} = \{\theta : (\theta, \Downarrow) = T\}\); the set \(S_{\Downarrow}\) is a closed subset of \(\square\). If \(\Downarrow \in C\), let \(S_{\Downarrow} = \bigcap\{S_{\theta} : \theta \in \Downarrow\}\); \(S_{\Downarrow}\), being an intersection of closed sets, is itself closed. The set \(S_{\Downarrow}\) consists of those truth assignments that satisfy all the sentences in \(\Downarrow\). Therefore \(\Downarrow\) is not satisfiable \(\Downarrow \in \square\). Thus the finite intersection property for the set of truth assignments states that if \(\Downarrow \in C\) is not satisfiable, there is a finite subset \(\Downarrow_0\) of \(\Downarrow\) that is also not satisfiable.

Note that this proof of the compactness theorem does not require that the either the alphabet in our language or set of sentences be countable.