2.2.1. Show that

(a) $\Gamma; \alpha \models \varphi$ if $\Gamma \models (\alpha \rightarrow \varphi)$. [The Exportation Rule]

First suppose $\Gamma; \alpha \models \varphi$; we claim that $\Gamma \models (\alpha \rightarrow \varphi)$. That is, it is asserted that for any structure $\mathcal{A}$ and valuation $s$ such that $\mathcal{A}$ satisfies every member of $\Gamma$ with $s$, then also $\mathcal{A}$ satisfies the formula $(\alpha \rightarrow \varphi)$ with $s$. Symbolically, if

$$\models_{\mathcal{A}} \gamma[s] \text{ for every } \gamma \in \Gamma,$$

then $\models_{\mathcal{A}} (\alpha \rightarrow \varphi)[s]$. Suppose $\mathcal{A}$ and $s$ satisfy the antecedent condition. In order to demonstrate $\models_{\mathcal{A}} (\alpha \rightarrow \varphi)[s]$, it is sufficient to show that $\models_{\mathcal{A}} \varphi[s]$ whenever $\models_{\mathcal{A}} \alpha[s]$. But if $\models_{\mathcal{A}} \alpha[s]$, then $\mathcal{A}$ and $s$ satisfy both every member of $\Gamma$ and $\alpha$. Since it is assumed that $\Gamma; \alpha \models \varphi$ (which asserts that every $\mathcal{A}$ and $s$ which satisfies $\Gamma; \alpha$ also satisfies $\varphi$), it follows that $\models_{\mathcal{A}} \varphi[s]$.

Now suppose that $\Gamma \models (\alpha \rightarrow \varphi)$; we claim that $\Gamma; \alpha \models \varphi$. That is, it is claimed that if $(\mathcal{A}, s)$ satisfies (1) every formula in $\Gamma$ and (2) the formula $\alpha$, then it satisfies $\varphi$. But if $(\mathcal{A}, s)$ satisfies $\Gamma$, then it satisfies $(\alpha \rightarrow \varphi)$. And if $(\mathcal{A}, s)$ also satisfies $\alpha$, then this, coupled with the fact that it satisfies $(\alpha \rightarrow \varphi)$, implies that it satisfies $\varphi$, which is what was to be proven.

(b) $\varphi \models \psi$ iff $\models (\varphi \leftrightarrow \psi)$.

Suppose $\varphi \models \psi$; this means that any $(\mathcal{A}, s)$ which satisfies $\varphi$ also satisfies $\psi$, and vice versa. We must then show that $\models (\varphi \leftrightarrow \psi)$; that is, that any $(\mathcal{A}, s)$ satisfies $(\varphi \leftrightarrow \psi)$. But $(\mathcal{A}, s)$ satisfies $(\varphi \leftrightarrow \psi)$ if it satisfies both $\varphi$ and $\psi$, or neither, which is what has been assumed.

Conversely, suppose that $\models (\varphi \leftrightarrow \psi)$; we claim that $\varphi \models \psi$. Thus we must show that any $(\mathcal{A}, s)$ pair which satisfies $\varphi$ satisfies $\psi$; and vice versa. But if $(\mathcal{A}, s)$ satisfies $\varphi$, then, since it is assumed that $\models (\varphi \leftrightarrow \psi)$, it follows that $(\mathcal{A}, s)$ satisfies $\psi$. Similarly, if $(\mathcal{A}, s)$ satisfies $\psi$, $(\mathcal{A}, s)$ satisfies $\varphi$.

2.2.2. Show that no one of the following sentences is logically implied by the other two.

(a) $\forall x \forall y \forall z (P_{xy} \rightarrow P_{yz} \rightarrow P_{xz})$.

(b) $\forall x \forall y (P_{xy} \rightarrow P_{yx} \rightarrow x = y)$.

(c) $\forall x \exists y P_{xy} \rightarrow \exists y \forall x P_{xy}$.

In order to do (a) and (b), we must find structures in which transitivity and antisymmetry fail to hold. There are lots of choices; here are two.
1. $\mathfrak{A} = (\mathbb{R}, P^\mathfrak{A} ab \text{ iff } ab > 0)$. Then (a) is true in $\mathfrak{A}$ (if $xy > 0$ and $yz > 0$, then $xz > 0$), and (c) is true in $\mathfrak{A}$ (because the antecedent is false: if $x = 0$ then there does not exist a $y$ such that $xy > 0$). However (b) is false: it states that if $xy > 0$ and $yx > 0$, then $x = y$.

2. $\mathfrak{A} = (\mathbb{R}, P^\mathfrak{A} if \text{iff } a = b^2)$. Then (a) is clearly false: $16 = 4^2$ and $4 = 2^2$, but of course $16 \neq 2^2$. On the other hand, both (b) and (c) are true. First we show (b) is true in $\mathfrak{A}$. Namely, if $a = b^2$ and $b = a^2$, then $a = a^4$, and it follows that $a = 0, 1$ (why?). But if $a = 0$, then $b^2 = 0$, hence $b = 0$; while if $a = 1$, then $b = a^2 = 1^2 = 1$. Finally, we show that (c) is true in $\mathfrak{A}$. This is easy; the antecedent is false for $x = -1$.

3. To show that (c) is not logically implied by (a) and (b), we must find a structure in which a predicate relation is both transitive and antisymmetric, but (c) fails. This is easy: $\mathfrak{A} = (\mathbb{R}, P^\mathfrak{A} ab \text{ iff } a \leq b)$. Then (a) and (b) are both true, while (c) is false since the antecedent is true ($x \leq x$ for all $x$) but the consequent is false (since there is no largest number).

2.2.3. Show that $\{\forall x(\alpha \rightarrow \beta), \forall x \alpha\} \models \forall x \beta$.

Suppose $(\mathfrak{A}, s)$ satisfies both $\forall x \alpha$ and $\forall x(\alpha \rightarrow \beta)$; we claim that $(\mathfrak{A}, s)$ also satisfy $\forall x \beta$. Let us understand what is being assumed and what we need to prove.

1. By definition, $(\mathfrak{A}, s)$ satisfies the formula $\forall x \alpha$ (that is, $\models_{\mathfrak{A}} \forall x \alpha[s]$), iff for every $d \in |\mathfrak{A}|$, $(\mathfrak{A}, s(x|d))$ satisfies $\alpha$, that is, iff $\models_{\mathfrak{A}} \alpha[s(x|d)]$.

2. Similarly, $(\mathfrak{A}, s)$ satisfies the formula $\forall x(\alpha \rightarrow \beta)$ iff for every $d \in |\mathfrak{A}|$, $(\mathfrak{A}, s(x|d))$ satisfies $(\alpha \rightarrow \beta)$.

What we need to show is that such a $(\mathfrak{A}, s)$ pair also satisfies the formula $\forall x \beta$; that is, that for every $d \in |\mathfrak{A}|$, $(\mathfrak{A}, s(x|d))$ satisfies $\beta$. But it follows from the hypothesis that $(\mathfrak{A}, s(x|d))$ satisfies both $\alpha$ and $(\alpha \rightarrow \beta)$, hence it must satisfy $\beta$ (why?).