Math 220 SELF PLACEMENT TEST

Instructions:

This test is very much like a final exam in Math 220, the first quarter of the basic sequence of calculus at Northwestern University. It covers topics in differential calculus. If you can pass it, you would be well advised to begin your math courses with Math 224 or a later course in the basic sequence. Work out as much of this exam as you can. Give yourself up to 90 minutes. Work carefully, writing out your work and stating your answers clearly, just as if this was an ordinary (not self-placement) exam. Do not look at the answers until you are finished.

Grade the exam following the attached grading scheme. If you make a simple numerical error, but judge your method to be correct, you may allot yourself some partial credits. If your score is higher than 75 (out of 110) with a minimum of 60 points (out of 80) on **Part I** , you have passed. If your total score is between 65 and 75, you have nearly passed. You can either retake 220 or start 224. However in your decision, please bear in mind that you would have done better on the test if you had reviewed the materials. If you are still in doubt, consult the Director of Calculus in the Mathematics Department.

Topic List:

Exponential, logarithmic and trigonometric functions

Limits; L'Hospital's rule

Derivatives; product, quotient and chain rule; implicit differentiation

Maximum and minimum values; optimization

Related rates, linear approximation, Newton's method

Part I (80 points)

1. (40 pts.) Compute the derivative of each of the following functions.(Do not simplify your answers.)

   (a) $f(x) = 4x^5 + \sqrt[5]{5x} + \arctan x + \pi$

   (b) $f(u) = \sin(2u)\ e^{-u}$

   (c) $h(t) = \dfrac{\cos(2t)}{1+t^2}$

   (d) $g(t) = \ln(2t^3 - 1)$

   (e) $f(x) = \sqrt[3]{\sin(x^2+1)}$

2. (10 pts.) Find the maximum and minimum value of $f(x) = x^3 - 6x^2 + 9x - 5$ on the closed interval $0 \le x \le 2$.

3. (15 pts.) Consider the function $f(x) = 3x^5 - 5x^3$. Find: The local maximum points and the local minimum points of the graph of $f(x)$. The intervals where the function is concave upward, those where it is concave downward and the inflection points. Use all this information to sketch the graph of f.

4. (15 pts.) Suppose you have to make a cylindrical can of volume $25\pi ft^3$. The top and the bottom circles are to be made of cardboard which costs $\$0.5/ft^2$ and the side is made by bending a rectangular piece of metal sheet which costs $\$5/ft^2$. What should be the dimensions of the cylinder if you want to minimize the cost? (Justify why your answer is a minimum.)

Part II (30 points)

5. (6 pts.) Find an equation of the tangent line to the curve $x^3 - y + y^2 = 3$ at the point (1,2).

6. (6 pts.) Find the following limits:

   (a) $\displaystyle\lim_{x\to -1^-} \frac{x^2-1}{3x^2-3x-6}$ (b) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$

7. (6 pts.) A ladder 10ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

8. (6 pts.) An astronaut is standing on a platform 3 meters above the moon's surface and throws a rock directly upward with an initial velocity of 16 m/s. Given the acceleration due to gravity on the moon's surface is -1.6 m/s^2, how high will the rock travel?

9. (6 pts.) Use linear approximation to estimate the value of $\sqrt{24}$.

ANSWERS TO SELFPLACEMENT TEST

Part I (80 points)

1. (40 pts., 8 pts each)

(a) $f'(x) = 20x^4 + (5x)^{-4/5} + \frac{1}{x^2+1}$ (chain rule for $\sqrt[5]{5x}; d/dx\sqrt[5]{5x} = 5 \cdot \frac{1}{5}(5x)^{-4/5}$; the derivative of π is zero)

(b) $f'(u) = 2\cos(2u)\ e^{-u} - \sin(2u)\ e^{-u}$ (product rule)

(c) $h'(t) = \dfrac{-2\sin(2t)(1+t^2) - 2t\cos(2t)}{(1+t^2)^2}$ (quotient rule)

(d) $g'(t) = \dfrac{6t^2}{(2t^3-1)}$ (chain rule)

(e) $f(x) = \dfrac{2}{3}x(\sin(x^2+1))^{-2/3}\cos(x^2+1)$ (chain rule)

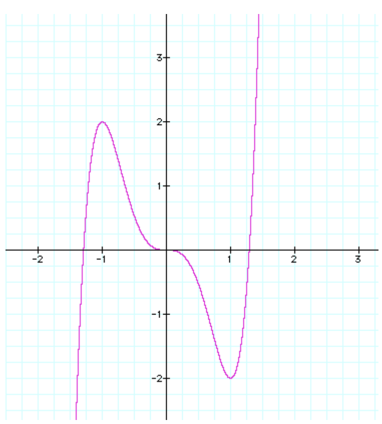
2. (10 pts.) $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ Critical points are $x = 1$ and the two endpoints $x = 0$ and $x = 2$. Note that $x = 3$ is not a critical point since it is not in the domain. Compare critical values: $f(0) = -5, f(1) = -1$ and $f(2) = -3$. Therefore the maximum value is -1 at $x = 1$ and the minimum value is -5 at $x = 0$.

3. (15 pts.) $f'(x) = 15x^2(x+1)(x-1), f"(x) = 30x(2x^2-1)$

f is increasing on $(-\infty, -1)$ and $(1, \infty)$, decreasing on $(-1, 1)$;

f is concave up $(-\sqrt{\frac{1}{2}}, 0)$ and $(\sqrt{\frac{1}{2}}, \infty)$, and concave down on $(-\infty, -\sqrt{\frac{1}{2}})$ and $(0, \sqrt{\frac{1}{2}})$. The graph of f has a local maximum at $x = -1$ and a local minimum at $x = 1$. f has inflection points at $x = -\sqrt{\frac{1}{2}}, x = 0$ and $x = \sqrt{\frac{1}{2}}$.

The graph of f:



4. (15 pts.) Volume $= 25\pi = \pi r^2 h$, thus $h = 25/r^2$.

The cost $C = 250\pi rh + \pi r^2 = 250\pi/r + \pi r^2, r > 0$.

$C'(r) = -250\pi/r^2 + 2\pi r = 0$ for $r = 5$.

The cost function C is decreasing on $(0, 5)$ and increasing on $(5, \infty)$, thus the minimum cost is obtained when $r = 5$. The dimensions of the cylinder that minimize the cost are radius 5 ft and height 1 ft.

Part II (30 points)

5. (6 pts.) $\frac{dy}{dx} = -1$ and the equation of the tangent line through $(1,2)$ with slope -1 is $y = -x + 3$.

6. (6 pts.) (a) $\lim_{x \to -1^-} \frac{x^2 - 1}{3x^2 - 3x - 6} = \frac{2}{9}$ (b) $\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$ (L'Hospital)

7. (6 pts.) The top of the ladder is sliding down the wall at a rate of $-3/4$ ft/s when the bottom of the ladder is 6 ft from the wall. (Related Rates)

8. (6 pts.) Solve the initial value problem $a = -1.6$ m/s^2,$v_0 = 16$ m/s, $s_0 = 3$ ft . Thus $v = -1.6t + 16, s = -0.8t^2 + 16t + 3$. The rock reaches the highest point when $v = 0$ after $t = 10$ seconds and the height then is $s(10) = 83$ ft.

9. (6 pts.) $f(x) = \sqrt{x}, a = 25; L(x) = 5{+}1/10(x{-}25)$ and $\sqrt{24} \approx 5 + 1/10(-1) = 4.9$.