

ALGEBRA FALL 2026 PRELIMINARY EXAM

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such. If you attempt more than three problems in one part, then the three problems with highest scores will count.

PART 1

- (1) Let k be a field, and define a functor on the category of finite-dimensional vector spaces

$$F : \text{Vect}_k^{\text{fd}} \longrightarrow \text{Vect}_k^{\text{fd}}, \quad F(V) = V \otimes_k V,$$

with $F(f) = f \otimes f$. For each V , let

$$\tau_V : V \otimes_k V \longrightarrow V \otimes_k V, \quad \tau_V(v \otimes w) = w \otimes v.$$

- (a) Show that $\tau : F \Longrightarrow F$ is a natural transformation.
- (b) Show that every natural transformation $T : F \Longrightarrow F$ is uniquely determined by its effect on $E = k^2$.
- (c) Prove that every natural transformation $T : F \Longrightarrow F$ has the form

$$T = a \text{id}_F + b\tau$$

for unique $a, b \in k$. (*Hint:* Think about naturality of the two coordinate projections $E \rightarrow E$)

- (2) Let V be an n -dimensional vector space over a field k , and let $0 \leq r \leq n$.
- (a) Show that for every nonzero $\alpha \in \Lambda^r(V)$ there is some $\beta \in \Lambda^{n-r}(V)$ such that $\alpha \wedge \beta \neq 0$.
 - (b) Show that the map

$$\Phi : \Lambda^r(V) \longrightarrow \text{Hom}_k(\Lambda^{n-r}(V), \Lambda^n(V)), \quad \Phi(\alpha)(\beta) = \alpha \wedge \beta,$$

is an isomorphism.

- (c) Fix a nonzero $\omega \in \Lambda^n(V)$ and define $\langle \alpha, \beta \rangle \in k$ by

$$\alpha \wedge \beta = \langle \alpha, \beta \rangle \omega.$$

If $A \in \text{GL}(V)$, show that

$$\langle \Lambda^r(A)\alpha, \beta \rangle = \det(A) \langle \alpha, \Lambda^{n-r}(A^{-1})\beta \rangle.$$

- (3) Let $A \in M_4(k)$ have characteristic polynomial

$$\chi_A(x) = (x^2 + x + 2)^2.$$

Find a complete set of representatives for the similarity classes of such matrices when

- (a) $k = \mathbb{Q}$;
- (b) $k = \mathbb{F}_2$.

- (4) Let G be a finite group. Recall that the *upper central series* of G is defined recursively by

$$Z_0(G) = \{1\}, \quad Z_{j+1}(G)/Z_j(G) = Z(G/Z_j(G)).$$

The group G is called *nilpotent* if $Z_c(G) = G$ for some $c \geq 0$.

Let $p_1, \dots, p_r$ be the distinct prime divisors of $|G|$, and for each i let P_i be a Sylow p_i -subgroup of G .

- (a) Suppose that G is nilpotent. Show that every P_i is normal in G .
 (b) Conversely, suppose that every P_i is normal in G . Show that the multiplication map

$$P_1 \times \cdots \times P_r \longrightarrow G, \quad (g_1, \dots, g_r) \longmapsto g_1 \cdots g_r,$$

is an isomorphism. Deduce that G is nilpotent.

- (5) Let $R = k[x, y]$, where k is a field. Consider the R -module $k \cong R/(x, y)$.
 (a) Show that the complex

$$0 \longrightarrow R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \longrightarrow k \longrightarrow 0,$$

where

$$d_2(c) = (-yc, xc) \quad \text{and} \quad d_1(a, b) = xa + yb,$$

is exact.

- (b) Compute $\text{Tor}_i^R(k, k)$ and $\text{Ext}_R^i(k, k)$ for every $i \geq 0$.

PART 2

- (1) Let $f(X) \in F[X]$ be irreducible of degree n , and let E/F be a field extension of degree m . Assume that $\gcd(m, n) = 1$. Prove that $f(X)$ remains irreducible in $E[X]$.

- (2) Let F be a field of characteristic $p > 0$, and let $f(X) \in F[X]$ be irreducible.
 (a) Prove that there exist an integer $e \geq 0$ and an irreducible separable polynomial $g(X) \in F[X]$ such that

$$f(X) = g(X^{p^e}).$$

- (b) Deduce that every root of $f(X)$ has multiplicity p^e in any splitting field of f .

- (3) Let E/F be a finite Galois extension with Galois group G , let $H \leq G$, and let $L = E^H$ be the fixed field of H . If $N_G(H)$ denotes the normalizer of H in G , prove that

$$\text{Aut}(L/F) \cong N_G(H)/H.$$

- (4) Determine the Galois group over $\mathbb{Q}$ of

$$f(X) = X^4 - 2X^3 - 8X - 3.$$

Give the group both up to isomorphism and as a permutation group on the roots of f .

- (5) (a) Prove that all rational solutions $(a, b) \in \mathbb{Q}^2$ of $a^2 + b^2 = 1$ are of the form

$$a = \frac{s^2 - t^2}{s^2 + t^2}, \quad b = \frac{2st}{s^2 + t^2},$$

for some $s, t \in \mathbb{Q}$, not both zero.

- (b) Deduce that, after possibly interchanging the two legs, every right triangle with integer side lengths has side lengths

$$d(m^2 - n^2, 2mn, m^2 + n^2)$$

for some integers d, m, n .

Hint: Apply Hilbert's Theorem 90 to $\mathbb{Q}(i)/\mathbb{Q}$.

PART 3

For multi-part questions, you may use the results of earlier parts in later steps without penalty, regardless of whether you completed the earlier proof.

(1) Let

$$R = \mathbb{C}[x, y]/(y^2 - x^2(x + 1)).$$

- (a) Prove that R is an integral domain.
- (b) Show that the element

$$\frac{y}{x} \in \text{Frac}(R)$$

is integral over R .

- (c) Determine the integral closure of R in its field of fractions.

(2) Let R be a ring and let $f : M \rightarrow M$ be a surjective module homomorphism.

- (a) Suppose that $\ker f^{n+1} = \ker f^n$ for some $n \geq 1$. Prove that f is an isomorphism.
- (b) Drop the assumption in part (a). Suppose that R is a finitely generated $\mathbb{Q}$ -algebra, and that M is a finitely generated R -module. Prove that f must be an isomorphism.

(3) Show that every non-zero minimal prime ideal of $\mathbb{Z}[x, y, z]$ is principal.

(4) Let $A \subset B$ be rings and suppose that B is integral over A . Show that $J(B) \cap A = J(A)$, where $J(B)$ and $J(A)$ are the Jacobson radicals of B and A respectively.

(5) Consider the inclusion $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, y]$, and let $\pi : \text{Spec } \mathbb{C}[x, y] \longrightarrow \text{Spec } \mathbb{C}[x]$ be the induced map.

- (a) Describe the fiber $\pi^{-1}(\mathfrak{p})$ for $\mathfrak{p} \in \text{Spec } \mathbb{C}[x]$.
- (b) Let $Z = V(xy - 1) \subseteq \text{Spec } \mathbb{C}[x, y]$. Determine $\pi(Z)$.