ALGEBRA PRELIMINARY EXAM, JUNE 2019

INSTRUCTIONS: Do **three** problems from each part below. In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas, but if a problem asks you to state or prove one such, you need to provide the full details.

Part 1: Groups, rings and modules

(1) Show that every group of order 35 is cyclic.
(2) Describe the conjugacy classes of S_4 and A_4.
(3) Let R be an integral domain.
(a) If $a, b \in R$, with a a unit, show that the mapping $X \to aX + b$ extends to a unique automorphism of the polynomial ring $R[X]$. Find the inverse of this automorphism.
(b) Show that all automorphisms of $R[X]$ are of the form described in (a).
(4) Let d be a integer which is not a square, and $R = \mathbb{Z}[\sqrt{d}]$. For $x = m+n\sqrt{d} \in R$, where $m, n \in \mathbb{Z}$, define the "conjugate" of x by $\bar{x} = m - n\sqrt{d}$, and a function $N\colon R \to \{0, 1, 2, 3, \cdots\}$ by $N(x) = |x\bar{x}|$.
(a) Show that $N(1) = 1$, $\overline{xy} = \bar{x}\bar{y}$, and $N(xy) = N(x)N(y)$ for all $x, y \in R$.
(b) Show that $x \in R$ is a unit if and only if $N(x) = 1$.
(c) Suppose $x \in R$ and $N(x) = p$ is a prime integer. Show that x is an irreducible element of R, whereas p is not.
(d) Suppose p is a prime and $p \neq N(x)$ for all $x \in R$. Show that p is an irreducible element of R.
(5) Let M be a finitely generated module over an integral domain R. Show that if R is a PID, then M is torsion-free if and only if it is free. Prove that this last property characterizes PID's, in the following sense: show that R is a PID if and only if every submodule of R is free.

Part 2: Fields, Galois theory and representation theory

(1) Let $\zeta = \zeta_7$ be a primitive 7-th root of unity. Show that $\mathbb{Q}(\zeta)$ has a unique subextension K of degree 2 over $\mathbb{Q}$ and a unique subextension L of degree 3 over $\mathbb{Q}$. Show that $K = \mathbb{Q}(\gamma)$, with $\gamma = \zeta^4 + \zeta^2 + \zeta$.
(2) Let K be a finite field of order p^n, for some prime p.
(a) Show that K is normal over $\mathbb{F}_p$, i.e. the splitting field of a polynomial.
(b) Determine the Galois group of K over $\mathbb{F}_p$.
(3) Determine the Galois group of the polynomial $X^4 - 10X^2 + 20$ over $\mathbb{Q}$.
(4) Find the character table of A_4; include details.

(5) Let G be a finite group, and $\rho\colon G \to \mathrm{GL}(V)$ a faithful representation of G (i.e. ρ is injective) of dimension n over $\mathbb{C}$. Suppose that $|\chi(g)| = n$, where χ is the character of V. Prove that g is in the center of G.

Part 3: Linear and homological algebra

(1) Find all the similarity classes of matrices in $M_4(\mathbb{R})$ satisfying the equation
$$A^3 + A = A^2 + I_4.$$

(2) Prove the following version of the "four lemma": if
$$\begin{array}{ccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 \end{array}$$
is a commutative diagram of R-modules with exact rows, α is an epimorphism and β and δ are monomorphisms, then γ is a monomorphism.

(3) Let R be a commutative ring, I an ideal in R, and M an R-module. Show that
$$M \otimes_R R/I \simeq M/IM.$$
Deduce that if J is another ideal in R, then
$$R/I \otimes_R R/J \simeq R/I + J.$$

(4) Compute
$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/14\mathbb{Z}, \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z})$$
for all $i \geq 0$.

(5) Justify your answers to the questions below:
(a) Is $\mathbb{Q}/\mathbb{Z}$ an injective $\mathbb{Z}$-module?
(b) Is $\mathbb{Q}/\mathbb{Z}$ a projective $\mathbb{Z}$-module?
(c) Is a finite abelian group G an injective $\mathbb{Z}$-module?
(d) Is a finite abelian group G a projective $\mathbb{Z}$-module?