ALGEBRA PRELIMINARY EXAM, FALL 2024

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote standard theorems and formulas, unless a problem specifically asks you to justify such.

1. Groups, rings and modules

(1) Find an explicit isomorphism of groups between $\mathrm{GL}_2(\mathbb{F}_2)$ and S_3.

(2) Let $G = \mathrm{GL}_3(\mathbb{F}_2)$. You may use the fact that $|G| = 168$.
  (a) How many Sylow 3-subgroups does G have?
  (b) Find an element of order 7 in G. (You may find it useful to look at an action of G on a particular 7-element set).

(3) Let $\mathcal{G}$ be the category of groups. Let A be the identity functor on $\mathcal{G}$. Let B be the functor $G \mapsto G^{\mathrm{op}}$ (where $G \mapsto G^{\mathrm{op}}$ is the set G with the multiplication $(a, b) \mapsto ba$).
  (a) Find all natural transformations from A to A.
  (b) Find all natural transformations from A to B.

(4) (a) Let $A = F[x]$ where F is a field. Show that the map
$$\mathrm{Hom}_A(F[x, x^{-1}], F[x]) \to \mathrm{Hom}_A(F[x, x^{-1}], F[x]/(x-1))$$
induced by the projection $F[x] \to F[x]/(x-1)$ is not onto.
  (b) Show that a direct limit of projective modules is not necessarily projective.

(5) Let R be the ring defined by three generators x, y, q and three relations $yx = qxy$; $qx = xq$; $qy = yq$.
  (a) Show that the elements $x^k y^m q^n$, $k, m, n \geq 0$, form a basis of R over $\mathbb{Z}$.
  (b) Consider two left R-modules $M = R/Rx$ and $N = R/Ry$. Show that M and N are not isomorphic.

2. Linear algebra and Galois theory

(1) Find a complete set of representatives for the similarity classes of 4×4 matrices A satisfying $(A^2 + I_4)^2 = 0$ over the field
  (a) $\mathbb{Q}$.
  (b) $\mathbb{F}_2$.

(2) Let V be an n-dimensional $\mathbb{C}$-vector space and let $T : V \to V$ be a linear map such that $T^n = 0$ but $T^{n-1} \neq 0$.
  (a) Show that there is a basis $e_1, \cdots e_n$ of V such that $T(e_1) = 0$ and $T(e_i) = e_{i-1}$ for $1 < i \leq n$.
  (b) Suppose $S : V \to V$ is a linear map such that $ST = TS$. Show that for each $1 \leq i \leq n$, S preserves the span of $e_1, \cdots e_i$.

(3) Let f be an irreducible polynomial of degree 5 over a field K. Suppose α, β are two roots of f in an algebraic closure of K. Show that the degree

$[K(\alpha,\beta) : K]$ is either 5, 10, 15 or 20. Which among these values are possible when K is a finite field?

(4) Let $K = \mathbb{Q}(e^{2\pi i/9}) \subset \mathbb{C}$. Describe all the subfields of K, and for each subfield write down a primitive element over $\mathbb{Q}$. (You may use the irreducibility of the n-th cyclotomic polynomial).

(5) Determine the Galois group of the polynomial $x^4 + 7x^2 + 7$ over
 (a) $\mathbb{Q}$.
 (b) $\mathbb{F}_5$.

3. Commutative algebra

(1) Consider the ring $R = \mathbb{C}[x,y]/(y^2 - x^4)$
 (a) Construct a finite map of rings $\mathbb{C}[t] \to R$.
 (b) Find the Krull dimension $\dim R$.
 (c) Find the maximal ideal $\mathfrak{m}$ of R such that $R_{\mathfrak{m}}$ is **not** a regular local ring.

(2) Let A be an Artinian local ring. Show that $\prod_{i=1}^{\infty} A$ is a free A module. (Hint: lift a basis modulo a maximal ideal).

(3) Let k be a field and suppose R is a complete local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Show that R is Noetherian if and only if

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 < \infty$$

(You may assume without proof that the power series ring $k[\![x_1, \cdots x_n]\!]$ is Noetherian).

(4) Let $\{\mathfrak{p}_i | i \in I\}$ be a family of prime ideals of a commutative ring A. For a subset X of Spec$A$ denote by $\overline{X}$ the closure of X in the Zariski topology. Prove that

$$\cap_i \overline{\{\mathfrak{p}_i\}} = \emptyset$$

if and only if

$$\sum_i \mathfrak{p}_i = A$$

(5)
- Let S be a multiplicative subset of a commutative ring A. Construct a bijection between the following sets:
  (a) the set of ideals $\mathfrak{a}$ of A such that for any $s \in S$ and any $a \in A$, $sa \in \mathfrak{a}$ if and only if $a \in A$;
  (b) the set of ideals of the localized ring A_S.
- Let $\mathfrak{p}$ be a maximal element of the set of ideals of A that do not intersect with S. Show that $\mathfrak{p}$ is prime.