ALGEBRA PRELIMINARY EXAM JUNE 2025

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such.

Part 1: Groups, rings, modules, category theory

(1) Consider a short exact sequence of groups

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 1$$

(a) Show that if there is a homomorphism $p : G \to K$ such that $p \circ i = id_K$ then $G \cong H \times K$.
(b) Show that if there is a homomorphism $s : H \to G$ such that $\pi \circ s = id_H$ then $G \cong K \rtimes H$ is a semi-direct product of H and K with respect to some homomorphism $\varphi : H \to \mathrm{Aut}(K)$.

(2) Let $G = \mathrm{SL}_2(\mathbb{F}_3)$.
(a) Show that $|G| = 24$.
(b) Compute the center of G.
(c) Show that G is **not** isomorphic to S_4.

(3) Let G be a finite group.
(a) Show that G admits an injective homomorphism into the symmetric group S_n for some n.
(b) Show that there are no surjective group homomomorphism $S_n \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^3$ for any n.
(You may use the fact that S_n is generated by two elements).

(4) Let R be a ring. Suppose M is the cokernel of some R-linear map $f : R^m \to R^n$. Let $g : R^N \twoheadrightarrow M$ be a surjection.
(a) Construct a commutative diagram

$$\begin{tikzcd} R^{n+N} \arrow[r, "p"] \arrow[d, "q"] & R^N \arrow[d, "g"] \\ R^n \arrow[r, "\mathrm{coker}(f)"] & M \end{tikzcd}$$

such that p, q are surjective.
(b) Show that $\ker(g)$ is a finitely generated R-module.

(5) Let R be a ring, and let $id_R : R{-}\mathrm{mod} \to R{-}\mathrm{mod}$ be the identity functor which sends each R-module M to itself. Let T be a natural transformation $T : id_R \implies id_R$. Show that there is an element $x \in Z(R)$ in the center of R such that $T(M) : M \to M$ is the multiplication by x map for *all* R-modules M.

Part 2: Linear algebra and Galois theory

(1) The matrix A over a field k has characteristic polynomial

$$P_A(x) = (x+1)^3(x^2+1)$$

Find a complete set of representatives for the similarity classes of such A when
(a) $k = \mathbb{Q}$.
(b) $k = \mathbb{F}_2$.

(2) Let $T : \mathbb{Q}^n \to \mathbb{Q}^n$ be a $\mathbb{Q}$-linear endomorphism such that its minimal polynomial is $x^n - 1$. Let $S : \mathbb{Q}^n \to \mathbb{Q}^n$ be a $\mathbb{Q}$-linear endomorphism such that $S \circ T = T \circ S$. Show that there is a polynomial $h \in \mathbb{Q}[x]$ such that
$$S = h(T)$$

(3) (a) Suppose $f, g \in \mathbb{F}_p[x]$ be irreducible polynomials of the same degree. Let α, β be roots of f, g in $\overline{\mathbb{F}}_p$. Show that there is a polynomial $h \in \mathbb{F}_p[x]$ such that
$$\beta = h(\alpha)$$
(b) Suppose $\alpha, \beta \in \overline{\mathbb{F}}_2$ such that
$$\alpha^3 + \alpha + 1 = 0$$
$$\beta^3 + \beta^2 + 1 = 0$$
Find an explicit polynomial $h \in \mathbb{F}_2[x]$ such that $\beta = h(\alpha)$.

(4) Let $K/\mathbb{Q}$ be a normal extension such that $[K : \mathbb{Q}] = 2^n$. Show that we can find $\alpha_1, \cdots \alpha_n \in K$ such that $K = \mathbb{Q}(\alpha_1, \cdots \alpha_n)$ and
$$\alpha_i^2 \in \mathbb{Q}(\alpha_1, \cdots \alpha_{i-1})$$
for each $1 \leq i \leq n$.

(5) Let K be the normal closure of $k(\sqrt{3+\sqrt{3}})$ for some field k. Determine the Galois group $\mathrm{Gal}(K/k)$ when
(a) $k = \mathbb{Q}$.
(b) $k = \mathbb{F}_5$.

Part 3: Commutative algebra and Homological algebra

(1) Let $f(x) = x^{2g+1} + a_{2g}x^{2g} + \ldots a_1 x + a_0$ be a polynomial with complex coefficients. Prove that $\mathbb{C}[x,y]/(y^2 - f(x))$ is an integral domain. Further, give a full characterization (with proof) of when $\mathbb{C}[x,y]/(y^2 - f(x))$ is integrally closed in its field of fractions.

(2) Let R be a noetherian ring, and let M be an R-module. Prove that M has a submodule isomorphic to $R/\mathfrak{p}$, where $\mathfrak{p} \subset R$ is a prime ideal.

(3) (a) Suppose that $0 \to M' \to M \to M'' \to 0$ is exact with M'' flat. Prove that M is flat if and only if M' is flat.
(b) Prove that $\mathrm{Ext}^i_{\mathbb{Z}}(A, B) = 0$ for $i > 1$ where A and B are two finitely generated abelian groups.

(4) Let $R = K[x,y,w,z]/(y^2 - xz, w^3 + y^2 - x^2 - xz)$.
(a) Show that R is an integral domain.
(b) Find an inclusion $A \subset R$ where A is a polynomial ring with coefficients in K and R is a finite A-algebra.
(c) Find the Krull dimension of R.

(5) Recall that a topological space X is said to be noetherian if every descending chain of closed subsets stabilizes. i.e. $X_1 \supset X_2 \ldots\ldots$ where each $X_i \subset X$ is closed eventually becomes constant. Find a proof or a counterexample for each statement below. If you find a counter-example, you need to prove that it is indeed a counter-example.

(a) If R is a Noetherian ring, then Spec R is a Noetherian space.

(b) If Spec R is a Noetherian space, then R is a Noetherian ring.