PRELIMINARY EXAM IN ANALYSIS SPRING 2019

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State the Dominated Convergence Theorem.
(b) Let $\{f_n\}$ be a sequence of integrable functions with

$$\int |f_n - f_{n-1}| d\mu \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty.$$

Show that f_n converges almost everywhere and

$$\int \left(\lim_n f_n\right) d\mu = \lim_n \int f_n \, d\mu.$$

(2) Suppose $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. For $h \in \mathbb{R}^d$, set $f^h(x) = f(h+x)$.
(a) Show that $\|f^h - f\|_p \to 0$ as $h \to 0$.
(b) Prove that if $f \in L^1(\mathbb{R}^d)$, then

$$\int f(x) e^{-ix \cdot z} dx \to 0$$

as $|z| \to \infty$.

(3) Let

$$\mathcal{A} = \left\{ f \in L^2([0,1]) : \int_0^1 f = 3, \int_0^1 x f(x) = 2 \right\}.$$

Find

$$\min_{f \in \mathcal{A}} \|f\|_2,$$

and a function for which the minimum is attained. *Hint: Consider functions of the form $g(x) = (x + \lambda) f(x)$ with $\lambda \in \mathbb{R}$ and $f \in \mathcal{A}$.*

(4) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. For a real valued measurable function f set

$$r(f) = \int \frac{|f|}{1 + |f|} d\mu.$$

Show that a sequence of (f_n) of integrable functions converges in measure to f if and only if $r(f_n - f) \to 0$.

(5) (a) Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be of bounded variation.
(b) Suppose that $f : [a, b] \to \mathbb{R}$ is a function of bounded variation. Let

$$g(a) = 0, \quad g(x) = \frac{1}{x-a} \int_a^x f(t)dt, \quad x \in (a, b].$$

Show that g is of bounded variation on $[a, b]$.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let H be a Hilbert space and $A : H \to H$ a linear operator defined everywhere on H. Suppose that $(Ax, y) = (x, Ay)$ for all $x, y \in H$. Show that A must be bounded.

(2) Let $L^2(\mathbb{R}_+)$ be the Hilbert space of square integrable functions on $\mathbb{R}_+ = [0, \infty)$ (with respect to the Lebesgue measure). Define the operator

$$Tf(x) = \frac{1}{x} \int_0^x f(y)\, dy.$$

Show that $T : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is bounded but not compact.

(3) Let $p : \mathbb{R} \to \mathbb{R}_+$ be a nonnegative function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} p(x)\, dx = 1.$$

Define $p_h(x) = h^{-1}p(h^{-1}x)$ for $h > 0$. Let $f \in L^q(\mathbb{R})$ $(1 \le q < \infty)$ be a L^q integrable function and $f_h = f * p_h$ the convolution of f with p_h. Show the following two assertions: (1) $\|f_h\|_q \le \|f\|_q$; (2) $\lim_{h\to 0} \|f_h - f\|_q = 0$.

(4) Let $\mathbb{S}^1$ be the unit circle and define the Fourier coefficients of a function on $\mathbb{S}^1$ by

$$\hat{f}(n) = \int_{\mathbb{S}^1} f(x) e^{inx}\, dx.$$

Define the Sobolev space $H^s(\mathbb{S}^1)$ to be the space of integrable function f on $\mathbb{S}^1$ such that

$$\|f\|_s^2 = \sum_{n\in\mathbb{Z}} (1 + n^2)^s |\hat{f}(n)|^2 < \infty.$$

Let $s < t$. Show that the embedding $i_{s,t} : H^t(\mathbb{S}^1) \to H^s(\mathbb{S}^1)$ is compact.

(5) Suppose that $s > 0$. Show that for any $g \in H^s(\mathbb{R}^n)$ there is an $f \in H^{s+1/2}(\mathbb{R}^{n+1})$ such that $f(x_1, \cdots, x_n, 0) = g(x_1, \cdots, x_n)$ and $\|f\|_{s+1/2} \le C\|g\|_s$. Here C is a constant independent of g.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) (a) State Morera's theorem.

(b) Let $U \subseteq \mathbf{C}$ be open and convex, and $f_n : U \to \mathbf{C}$ a sequence of holomorphic functions converging on compact subsets of U to a function f. Prove that f is holomorphic on U.

(c) Prove that $f_n' \to f'$ uniformly on compact subsets of U.

(2) Let $f : D \to D$ be a holomorphic map of the unit disk $D := \{z : |z| < 1\}$ into itself.

(a) Give automorphisms g and h of D such that $g(0) = a$, and $h(f(a)) = 0$.

(b) Prove that for all $a \in D$ one has

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

[*Hint*: Let $F = h \circ f \circ g$; apply Schwarz.]

(3) (a) State the Casorati-Weierstrass theorem.

(b) Let H be the upper half-plane $\{z : \Im z > 0\}$, and f a function analytic on $\overline{H}$ such that for some $A > 0$ and $\epsilon > 0$ one has

$$|f(\zeta)| \leq \frac{A}{|\zeta|^\epsilon}.$$

for all ζ. Prove that for any $z \in H$, one has the integral formula (the *Cauchy transform*)

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z}\, dt.$$

(4) (a) State *Liouville's theorem* for bounded entire functions.

(b) Prove that if $f : \mathbf{C} \to \mathbf{C}$ is a non-constant analytic function, then the image of f is dense in $\mathbf{C}$.

(5) Let f be analytic on the closed unit disk. Assume that $|f(z)| = 1$ if $|z| = 1$ and f is not constant. Prove that the image of f contains the unit disk.