PRELIMINARY EXAM IN ANALYSIS FALL 2019

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State Fatou's lemma and the dominated convergence theorem.
(b) Show that the inequality in Fatou's lemma may be strict.
(c) Using Fatou's Lemma, prove the dominated convergence theorem.

(2) State Fubini's Theorem. Prove that if $f \in L^1(0,1)$ and $\alpha > 0$, then

$$F(x) = \int_0^x (x-t)^{\alpha-1} f(t)dt$$

exists for almost every $x \in (0,1)$ and $F \in L^1(0,1)$.

(3) Suppose μ_i and ν_i be finite measures on $(\Omega, \mathcal{F})$ with $\mu_i \ll \nu_i$ for $i = 1,2$. Let $\mu = \mu_1 \times \mu_2$ and $\nu = \nu_1 \times \nu_2$ be the corresponding product measures on $(\Omega \times \Omega,\ \mathcal{F} \times \mathcal{F})$.
(a) Show that $\mu \ll \nu$.
(b) Prove that $\frac{d\mu}{d\nu}(x,y) = \frac{d\mu_1}{d\nu_1}(x)\frac{d\mu_2}{d\nu_2}(y)$.

(4) Prove that if $f \in L^1[0,\pi]$ and

$$F(x) = \int_0^x f(t)\sin(t)dt$$

then F is differentiable for almost every x in $[0,\pi]$. Compute F'.

(5) Let $f \in L^p(\Omega, \mathcal{F}, \mu)$ be a nonnegative function. Show that for any $p \geq 1$,

$$\int_\Omega f(x)^p dx = p\int_0^\infty \lambda^{p-1}\mu\{f \geq \lambda\}.$$

Does the same result hold without the assumption that $f \geq 0$?

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $(X, \mathscr{F}, \mu)$ be a measure space with $\mu(X) = 1$. Show that for any measurabe function f, the function $p \mapsto \|f\|_p$ is nondecreasing in $p \geq 1$.

(2) Let A be a bounded, symmetric, and compact operator on a Hilbert space. Show that either $\|A\|$ or $-\|A\|$ is an eigenvalue of A.

(3) (a) Let H be a (linear) subspace of $L^2(\mathbb{R})$. Show that H is dense in $L^2(\mathbb{R})$ if

$$\int_{\mathbb{R}} h(x) f(x)\, dx = 0$$

for all $h \in H$ implies $f = 0$. (b) Let $h \in L^2(\mathbb{R})$ and define $h_t(x) = h(x+t)$. Show that the linear span of the translates $\{h_t,\ t \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$ if and only its Fourier transform $\widehat{h}$ is nonvanishing almost everywhere.

(4) Let $\mathbb{S}^1$ be the unit circle and define the Fourier coefficients of a function on $\mathbb{S}^1$ by

$$\hat{f}(n) = \int_{\mathbb{S}^1} f(x) e^{inx}\, dx.$$

Define the Sobolev space $H^s(\mathbb{S}^1)$ to be the space of integrable function f on $\mathbb{S}^1$ such that

$$\|f\|_s^2 = \sum_{n \in \mathbb{Z}} (1+n^2)^s |\hat{f}(n)|^2 < \infty.$$

Let $s < t$. Show that the embedding $i_{s,t} : H^t(\mathbb{S}^1) \to H^s(\mathbb{S}^1)$ is compact.

(5) Let $C_K^\infty(\mathbb{R})$ be the space of smooth functions on $\mathbb{R}$ with compact support. For each $f \in C_K^\infty(\mathbb{R})$ define

$$Hf(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\pi} \int_{|t|>\epsilon} \frac{f(x-t)}{t}\, dt.$$

(a) Show that the limit in the definition of Hf exists for all $f \in C_K^\infty(\mathbb{R})$. (b) Find the Fourier transform $\widehat{Hf}$ of Hf in terms of the Fourier transform $\widehat{f}$ of f. (c) Show that and $\|Hf\|_2 = \|f\|_2$ for all $f \in C_K^\infty(\mathbb{R})$.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Find all analytic isomorphisms of either (a) the complex plane $\mathbb{C}$ or (b) the open unit disk $D := \{z : |z| < 1\}$. (An analytic isomorphism of an open set U in the complex plane is a bijection f of U onto itself such both f and f^{-1} are analytic.) Whichever case you choose, be sure to give a complete proof.

(2) Find the Laurent expansion around 2 of the function

$$f(z) = \frac{z-1}{z(z-2)^3}$$

in the annulus $\{z : 0 < |z-2| < 2\}$.

(3) Let f_k, $k \geq 1$, be a sequence of analytic functions on a connected open set U converging to an analytic function f on U which does not vanish everywhere. *Prove*: If f has a zero of order n at a point $z_0 \in U$, then for every sufficiently small $\rho > 0$ and sufficiently large k (depending on ρ), the function f_k has exactly n zeros in the disk $D_\rho(z_0) := \{z : |z - z_0| < \rho\}$ (counting multiplicity). In addition, the zeros converge to z_0 as $k \to \infty$.

(4) Let $f : D \to D$ be an analytic function of the open unit disk D into itself. Prove that for all $a \in D$ one has

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

(*Hint*: Let g be an automorphism of D sending 0 to a, h an automorphism of D sending $f(a)$ to 0, and let $F := h \circ f \circ g$. Compute $F'(0)$ and apply Schwarz.)

(5) Let f be a function that is analytic inside and on the unit circle. Suppose that $|f(z) - z| < |z|$ on the unit circle.

(a) Show that $|f'(1/2)| \leq 8$.

(b) How many zeros does f have inside the unit circle?