PRELIMINARY EXAM IN ANALYSIS
SEPTEMBER 2026

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Show that

$$\lim_{n\to\infty} \int_0^\infty \frac{n\left(1-e^{-x/n}\right)}{x(1+x^2)}\,dx = \frac{\pi}{2}.$$

(2) Let $f \in L^1(\mathbb{R})$, $f \geq 0$, and for $t > 0$ define

$$\lambda_f(t) = m(\{x \in \mathbb{R} : f(x) > t\}),$$

where m is the Lebesgue measure. Prove that $\lambda_f(t) < \infty$ for every $t > 0$, that $t\lambda_f(t) \to 0$ as $t \to \infty$, and that

$$\int_0^\infty \lambda_f(t)\,dt = \int_{\mathbb{R}} f(x)\,dx.$$

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space. (a) Prove the Borel-Cantelli Lemma: If $E_1, E_2, \ldots \in \mathcal{M}$ satisfy

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty$$

then

$$\limsup_{n\to\infty} E_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$$

has measure zero. (b) Let f_n be measurable functions such that $f_n \to f$ in measure as $n \to \infty$. Prove that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that for almost every x,

$$f_{n_k}(x) \to f(x) \quad \text{as } k \to \infty.$$

(4) Let $p > 1$, and let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ which converge to zero in measure as $n \to \infty$ and

$$\sup_n \int_0^1 |f_n(x)|^p\,dx < \infty.$$

Show that

$$\int_0^1 |f_n(x)|\,dx \to 0, \quad \text{as } n \to \infty$$

and give an example to show that the result is false when $p = 1$.

(5) Let $[a,b] \subset \mathbb{R}$ be a compact interval and $f : [a,b] \to \mathbb{R}$ a function. (a) Define the total variation $T_f(a,b)$ of f on $[a,b]$. (b) Suppose that $T_f(a,b) < \infty$ (i.e. f has bounded variation). If we define

$$V(x) = T_f(a,x), \qquad a \le x \le b,$$

show that both V and $V - f$ are nondecreasing on $[a,b]$. (c) Conclude (briefly) that every function of bounded variation on $[a,b]$ is differentiable almost everywhere.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) For $1 \le p < \infty$, denote by $l^p(\mathbb{N})$ the Banach space of $\mathbb{C}$ valued sequences $x = \{x_n\}$ with the norm $\|x\|_p = \{\sum_{n=1}^{\infty} |x_n|^p\}^{1/p}$. For $p = \infty$, denote by $l^\infty(\mathbb{N})$ the Banach space of $\mathbb{C}$ valued sequences x with the norm $\|x\|_\infty = \sup_n |x_n|$. For $1 \le p \le \infty$, let $1 \le q \le \infty$ be the exponent dual to p defined by $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in l^p(\mathbb{N})$ and $y \in l^q(\mathbb{N})$ define

$$i_p(y)(x) = \langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

(a) For $1 \le p < \infty$, show that $y \mapsto i_p(y)$ is a conjugate-linear isometry between $l^q(\mathbb{N})$ and the dual space $l^p(\mathbb{N})^*$. (b) Show that the same map $y \mapsto i_\infty(y)$ is a conjugate-linear embedding from $l^1(\mathbb{N})$ into $l^\infty(\mathbb{N})^*$ which is not surjective.

(2) The Hardy transform of a function $f \in L^2[0,1]$ is defined by

$$Hf(x) = \frac{1}{x} \int_0^x f(y)\, dy.$$

(a) Show that H is a bounded operator from $L^2[0,1]$ into itself. (b) Is $H : L^2[0,1] \to L^2[0,1]$ compact? Justify your answer.

(3) Let X be a Banach space and $x_n \in X$ a sequence in X which converges weakly to $x \in X$. (a) Show that $\{x_n\}$ is bounded and

$$\|x\| \le \liminf_{n \to \infty} \|x_n\|.$$

(b) Show by an example that the inequality can be strict.

(4) Recall that $H^s(\mathbb{R})$ is the completion of the Schwartz space $\mathscr{S}(\mathbb{R})$ under the norm

$$\|f\|_{H^s} = \sqrt{\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \langle \xi \rangle^{2s}\, d\xi}, \qquad \langle \xi \rangle = \sqrt{1 + |\xi|^2}.$$

[$\hat{f}$ is the Fourier transform of f.] Define the trace operator $T : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$ by $Tf = f(0)$. Show that for any $s > 1/2$, the trace operator T can be extended to a bounded operator from $H^s(\mathbb{R})$ to $\mathbb{R}$.

(5) Suppose that $f \in \mathscr{S}(\mathbb{R})$ and define its shift by $f_t(x) = f(x+t)$. Show that the closure of the linear combinations of the shifts $\{f_t,\ t \in \mathbb{R}\}$ is the whole space $L^2(\mathbb{R})$ if the Fourier transform $\hat{f}$ does not vanish anywhere, i.e., $\hat{f}(\xi) \neq 0$ for every $\xi \in \mathbb{R}$.

Part III. Complex Analysis

In this part $D_r(w)$ denotes the open disc of radius r centered at w in the complex plane $\mathbb{C}$ and $\overline{D}_r(w)$ its closure.

Do **three** of the following five problems.

(1) Let $|a| < 1$ and let n be a positive integer. Show that the equation

$$(z-1)^n e^z = a$$

has n solutions in the right half-plane $\{\Re z > 0\}$, counted with multiplicity.

(2) Let H denote the space of holomorphic functions on the open unit disc $D_1(0)$ that are in $L^2(D_1(0); dx\,dy)$.

(a) Show that for each compact $K \subset D_1(0)$ there exists a C depending on K such that

$$\sup_{z\in K} |f(z)| \leq C\|f\|_{L^2(D_1(0))},\ f \in H.$$

(b) Show that H is a closed subspace of $L^2(D_1(0))$.

(3) Let $f : D_1(0) \to D_1(0)$ be holomorphic and satisfy

$$f(0) = f(a) = 0$$

for some $a \in D_1(0)\backslash\{0\}$. Show that

$$|f(z)| \leq |z| \left|\frac{a-z}{1-\bar{a}z}\right|$$

(4) Let f be an entire function that is not a polynomial. Show that for every $R > 0$, the image under f of $\mathbb{C}\backslash\overline{D}_R(0)$ is dense in $\mathbb{C}$. You may not quote the Casorati–Weierstrass theorem.

(5) Compute the integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^3}\,dx$$

by applying the residue theorem to a suitable branch of the function $\frac{\sqrt{z}}{1+z^3}$.