Geometry/Topology 440-1, 440-2, 440-3

April 2, 2025

440-1: FALL – Differentiable Manifolds

Text: Geometry of Manifolds by Richard Bishop and Richard Crittenden; Topology and Geometry by Glen Bredon

1. Differentiable manifolds: definition and examples (including projective spaces, Grassmannians, classical Lie groups, homogeneous spaces); submersion, immersion, and embedding (submanifolds)
2. Implicit function theorem; critical values and Sard's theorem
3. Tangent vectors as derivations; tangent space; basis in local coordinates; tangent vector fields and their flows
4. Lie brackets and their geometric meaning; distributions and integral manifolds; Frobenius's theorem
5. Cotangent vectors; tensors and contractions, definition of Lie derivative
6. Riemannian manifolds; arc length and Riemannian distance
7. The Levi-Civita connection; Christoffel symbols; covariant differentiation of tensor fields; parallel transport; examples
8. Variation of arc length; geodesics; exponential map; Gauss's lemma and local normal coordinates
9. Optional topics, not included in the prelim exam: Riemannian curvature tensor, correspondence between connected Lie subgroups and Lie subalgebras.

440-2: WINTER – Algebraic Topology

Text: Algebraic Topology by Allen Hatcher

1. Homotopy and the fundamental group
2. Covering spaces; fundamental theorem of algebra

3. The universal cover and the classification of covering spaces
4. Van Kampen's theorem, examples (Riemann surfaces, projective spaces, $K(G,1)$-spaces) and applications
5. Singular homology, homotopy invariance
6. Relative homology, long exact sequences, excision and Mayer–Vietoris
7. CW homology, degree, and Euler characteristic
8. Homological algebra, including the universal coefficient theorem and Künneth theorems
9. Basics of singular cohomology

440-3: SPRING – de Rham Cohomology

Text: Algebraic Topology by Allen Hatcher; Differential Forms in Algebraic Topology by Raoul Bott and Loring Tu; and Topology and Geometry by Glen Bredon

1. Differential forms and exterior differentiation; orientability and integration of differential forms on manifolds
2. Stokes's theorem; de Rham cohomology, connection to div-grad-curl
3. Sheaf cohomology and Čech cohomology
4. The de Rham theorem, the Poincaré lemma; cohomology with compact support
5. Orientability and Poincaré duality
6. Vector and fiber bundles; sphere bundles and Euler classes; Hopf maps
7. De Rham cohomology of classical Lie groups
8. Possible additional topics: Local systems and monodromy; Ehresmann's submersion theorem; Borel–Moore homology, other forms of Poincaré duality