GEOMETRY AND TOPOLOGY QUALIFYING EXAM
SEPTEMBER, 2019

All questions are of equal value. Your should solve six of the eight problems.

1) Calculate $H^3(\mathbb{CP}^n, \mathbb{Z})$.

2) The manifold $\mathrm{SL}(2, \mathbb{R})$ is the space of all 2×2 real matrices with determinant 1. Let $\mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R})$ be the subset of rotation matrices, that is, matrices $A \in \mathrm{SL}(2, \mathbb{R})$ such that $A^{-1} = A^T$ and $\det(A) = 1$. Show that there is a deformation retraction of $\mathrm{SL}(2, \mathbb{R})$ onto $\mathrm{SO}(2)$.

3) Let ξ and η be vector fields on a manifold M, and let $\mathcal{L}(\xi)$ and $\iota(\eta)$ be the operations on the de Rham complex $\Omega^*(M)$ of Lie derivative with respect to ξ and interior product with respect to η (sometimes denoted by $\mathcal{L}_\xi$ and $\eta \lrcorner$, respectively). Prove that

$$[\mathcal{L}(\xi), \iota(\eta)] = \iota([\xi, \eta]).$$

(Here the bracket on the left hand side denotes the commutator of operators.)

4) An almost-complex manifold is a manifold M whose tangent spaces are complex vector spaces smoothly. This may be expressed by giving an endomorphism $J \in \mathrm{Hom}(TM, TM)$ of the tangent bundle TM (a bundle map from TM to itself) such that $J^2 = -\mathrm{Id}$, so that a complex number $z = x + iy$ acts on TM by $x + yJ$.

   a) Show that the sphere S^2 is an almost-complex manifold.
   b) Show that an almost-complex manifold is orientable.
   c) Show that an orientable surface is an almost-complex manifold.

5) The real projective space $\mathbb{RP}^2$ is the quotient of the unit sphere in $\mathbb{R}^3$

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

by the involution

$$(x, y, z) \mapsto (-x, -y, -z).$$

Calculate its de Rham cohomology.

6) Let M be a manifold with boundary, N a manifold and let $f : M \to N$ be a smooth map. Show that there is a point $p \in N$ so that $f^{-1}(p)$ is a submanifold of M which is transverse to ∂M.

7) Let $\alpha \in \Omega^k M$ and $\beta \in \Omega^l M$. Suppose that α is closed, and that β is exact. Let N be a compact $(k+l)$-manifold with boundary, and suppose that $\int_N \alpha \wedge \beta \neq 0$. Show that there is no primitive $\lambda \in \Omega^{l-1} M$ of β, so that $\lambda|_{\partial N} = 0$.

8) Let $M \subseteq N$ be a submanifold, and let g be a Riemannian metric on N. Let ∇ be the Levi-Civita connection of g, and let ∇^M be defined by $\nabla^M_X Y = \pi(\nabla_X \tilde{Y})$, where X and Y are vector fields on M, $\tilde{Y}$ is an extension to N, and $\pi : TN_p \to TM_p$ is the orthogonal projection for $p \in M$.

   a) Show that ∇^M is well defined, and that it is the Levi-Civita connection for the restricted metric $g|_M$.
   b) For vector fields X and Y on M, let $\mathrm{I\!I}(X, Y) = \nabla_X \tilde{Y} - \nabla^M_X \tilde{Y}$ be the normal component of $\nabla_X \tilde{Y}$. Show that $\mathrm{I\!I}$ is symmetric in X and Y.