Instructions: Choose six of the eight problems to solve.

1. Let $n \geq 1$ be a natural number, and let $X$ be the union of the unit sphere in $\mathbb{R}^n$ and the line connecting the north and south poles.
   a) What is the fundamental group of this topological space?
   b) Let $n \geq 3$. Define a space $\tilde{X}$ and a map $\pi : \tilde{X} \to X$ which is the universal covering.

2. Consider the unit sphere in $\mathbb{C}^2$:
   $$S^3 = \{(w, z) \mid |w|^2 + |z|^2 = 1\}.$$  
   Given a positive integer $k$, consider the continuous map
   $$T : S^3 \to S^3$$
   given by the formula $T(w, z) = (e^{2\pi i/k}w, e^{2\pi i/k}z)$.
   a) Show that $T$ is a diffeomorphism.
   b) Let $\Gamma$ be the transformation group generated by $T$. Show that $T$ is a finite group. What is its order?
   c) Show that the quotient space $M = S^3/\Gamma$ is a manifold.
   d) What is the fundamental group of $M$?
   e) Classify the covering spaces of $M$.

3. Prove that if $\mathbb{CP}^{2n}$ is the universal cover of a smooth manifold $M$, then $M = \mathbb{CP}^{2n}$.

4. Recall the Poincare duality says that for a compact oriented manifold $M^n$, $H_i(M; \mathbb{Z})$ is isomorphic to $H^{n-i}(M; \mathbb{Z})$. Prove that if $M$ is a compact oriented manifold of dimension $2k$ and that $H_{k-1}(M; \mathbb{Z})$ is torsion free, i.e., torsion part is zero, prove that $H_k(M; \mathbb{Z})$ is also torsion free.

5. Let $M$ and $N$ be connected oriented $n$-dimensional differentiable manifolds. Their connected sum $M \# N$ is defined as follows: pick points $x$ and $y$ in $M$ and $N$ respectively, remove small open balls $B_\epsilon(x)$ and $B_\epsilon(y)$ around $x$ and $y$, and identify the boundaries of the resulting manifolds $M \setminus B_\epsilon(x)$ and $N \setminus B_\epsilon(y)$, which are diffeomorphic to the sphere $S^{n-1}$, by an orientation reversing diffeomorphism.
   Calculate the Euler characteristic of $M \# N$ in terms of $M$ and $N$. 
6. Let \( f : M \to S^1 \), and let \( N \subseteq M \) be a closed orientable 1–manifold.

a) Show that there is a \( \theta \in S^1 \), so that \( f^{-1}(\theta) \) is a submanifold, and \( f^{-1}(\theta) \cap N \).

b) Suppose that \( \Sigma \subseteq M \) is a compact orientable surface with boundary, so that \( \partial \Sigma = N \). Show that \( \int_N f^*(d\theta) = 0 \).

c) Let \( \theta \in S^1 \) be as in part (a), and suppose \( \int_N f^*(d\theta) = 0 \). Show \( f^{-1}(\theta) \cap N \) consists of an even number of points.

7. Let \( M \) be a smooth \( n \)-manifold equipped with a Riemannian metric, and let \( u : [0, 1] \times (-\varepsilon, \varepsilon) \to M \) be a smooth map so that the curve \( t \mapsto u(t, s_0) \) is a geodesic for all \( s_0 \in (-\varepsilon, \varepsilon) \). Let \( \gamma : [0, 1] \to M \) be the curve \( t \mapsto u(t, 0) \) and let \( V \) be the vector field along \( \gamma \) defined by

\[
V = \frac{\partial u}{\partial s} \bigg|_{s=0}.
\]

a) Show that \( V \) satisfies the Jacobi equation

\[
\frac{D^2 V}{dt^2} + R(V, \dot{\gamma})\dot{\gamma} = 0,
\]

where \( R(X, Y)Z \) is the curvature tensor of \( M \).

b) Show that the set of such \( V \) is a vector space of dimension \( 2n \).

8. Let \( M = \{(x, y, z) ; x^2 + y^2 = z^2, z > 0 \} \subseteq \mathbb{R}^3 \). Equip \( M \) with the Riemannian metric restricted from the Euclidean metric on \( \mathbb{R}^3 \). Show that \( M \) is locally Euclidean.