GEOMETRY AND TOPOLOGY QUALIFYING EXAM

Instructions: Choose six of the eight problems to solve.

1. Let $n \geq 1$ be a natural number, and let X be the union of the unit sphere in $\mathbb{R}^n$ and the line connecting the north and south poles.

a) What is the fundamental group of this topological space?
b) Let $n \geq 3$. Define a space $\tilde{X}$ and a map $\pi : \tilde{X} \to X$ which is the universal covering.

2. Consider the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{(w, z) \mid |w|^2 + |z|^2 = 1\}.$$

Given a positive integer k, consider the continuous map

$$T : S^3 \to S^3$$

given by the formula $T(w, z) = (e^{2\pi i/k} w, e^{2\pi i/k} z)$.

a) Show that T is a diffeomorphism.
b) Let Γ be the transformation group generated by T. Show that T is a finite group. What is its order?
c) Show that the quotient space $M = S^3/\Gamma$ is a manifold.
d) What is the fundamental group of M?
e) Classify the covering spaces of M.

3. Prove that if $\mathbb{CP}^{2n}$ is the universal cover of a smooth manifold M, then $M = \mathbb{CP}^{2n}$.

4. Recall the Poincare duality says that for a compact oriented manifold M^n, $H_i(M;\mathbb{Z})$ is isomorphic to $H^{n-i}(M;\mathbb{Z})$. Prove that if M is a compact oriented manifold of dimension $2k$ and that $H_{k-1}(M;\mathbb{Z})$ is torsion free, i.e., torsion part is zero, prove that $H_k(M;\mathbb{Z})$ is also torsion free.

5. Let M and N be connected oriented n-dimensional differentiable manifolds. Their connected sum $M\#N$ is defined as follows: pick points x and y in M and N respectively, remove small open balls $B_\epsilon(x)$ and $B_\epsilon(y)$ around x and y, and identify the boundaries of the resulting manifolds $M \setminus B_\epsilon(x)$ and $N \setminus B_\epsilon(y)$, which are diffeomorphic to the sphere S^{n-1}, by an orientation reversing diffeomorphism.

Calculate the Euler characteristic of $M\#N$ in terms of M and N.

6. Let $f : M \to S^1$, and let $N \subseteq M$ be a closed orientable 1–manifold.

a) Show that there is a $\theta \in S^1$, so that $f^{-1}(\theta)$ is a submanifold, and $f^{-1}(\theta) \pitchfork N$.
b) Suppose that $\Sigma \subseteq M$ is a compact orientable surface with boundary, so that $\partial\Sigma = N$. Show that $\int_N f^*(d\theta) = 0$.
c) Let $\theta \in S^1$ be as in part (a), and suppose $\int_N f^*(d\theta) = 0$. Show $f^{-1}(\theta) \cap N$ consists of an even number of points.

7. Let M be a smooth n-manifold equipped with a Riemannian metric, and let $u : [0, 1] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map so that the curve $t \mapsto u(t, s_0)$ is a geodesic for all $s_0 \in (-\varepsilon, \varepsilon)$. Let $\gamma : [0, 1] \to M$ be the curve $t \mapsto u(t, 0)$ and let V be the vector field along γ defined by

$$V = \left.\frac{\partial u}{\partial s}\right|_{s=0}.$$

a) Show that V satisfies the Jacobi equation

$$\frac{D^2 V}{dt^2} + R(V, \dot{\gamma})\dot{\gamma} = 0,$$

where $R(X, Y)Z$ is the curvature tensor of M.
b) Show that the set of such V is a vector space of dimension $2n$.

8. Let $M = \{(x, y, z); x^2 + y^2 = z^2, z > 0\} \subseteq \mathbb{R}^3$. Equip M with the Riemannian metric restricted from the Euclidean metric on $\mathbb{R}^3$. Show that M is locally Euclidean.