PRELIMINARY EXAM IN GEOMETRY SPRING 2024

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Differential Geometry

Do **three** of the following five problems.

(1) Let M be a closed manifold and let $f : M \to M$ be a smooth onto map. Prove that there is a nonempty open set $U \subseteq M$ and an integer k such that $f^{-1}(U)$ is diffeomorphic to $U \times \{1, \ldots, k\}$.

(2) Let M be a connected manifold and let $a, b \in M$. Prove that there is a diffeomorphism $f : M \to M$ such that $f(a) = b$.

(3) Let (M, g) be a connected and complete 2-dimensional Riemannian manifold. Prove that there are 3 points p_1, p_2, p_3 such that every isometry $\varphi : M \to M$ that fixes p_1, p_2, p_3 must be the identity.

(4) Let $S = \left\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\right\}$ and endow S with the Riemannian metric induced by the metric of $\mathbb{R}^3$. Using your favorite coordinate system:
   (a) compute the Levi–Civita connection on S;
   (b) compute the curvature of S.

(5) (a) Define what it means for a smooth manifold to be oriented.
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a submersion. Prove that $f^{-1}(0)$ is an orientable manifold.

Part II. Fundamental Group and Homology

Do **three** of the following five problems.

(1) Compute the fundamental groups of the following spaces.
   (a) $\mathrm{SO}(2)$
   (b) $\mathrm{SO}(3)$
   (c) $\mathrm{GL}(3, \mathbb{R})$

(2) Let G be a Lie group. Prove that $\pi_1(G)$ is abelian.

(3) Give a complete description of all covering spaces of the wedge $\mathbb{RP}^2 \vee \mathbb{RP}^2$.

(4) Using topology, prove that every subgroup of a free group is again free.

(5) Prove that $\mathbb{RP}^4$ does not admit a nowhere vanishing vector field.

Part III. Cohomology

Do **three** of the following five problems.

(1) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the double cover and let S be the inverse limit

$$S := \varprojlim\left(\cdots \xrightarrow{f} \mathbb{S}^1 \xrightarrow{f} \mathbb{S}^1 \xrightarrow{f} \mathbb{S}^1\right) .$$

  (a) Show that $H_1(S) = 0$.
  (b) Find a non-trivial element in $H^1(S)$ (or in $H^1_{dR}(S)$, if that's easier).

(2) Let M be the complement of the z axis in $\mathbb{R}^3$ and let $C = \{(x, y, 0) \mid x^2 + y^2 = 1\}$. Write down a closed and compactly-supported 2-form on M whose cohomology class is the Poincare dual of C.

(3) Let $f : S^1 \to S^1 \times S^1$ be $f(e^{i\theta}) = (e^{2i\theta}, e^{3i\theta})$. The image of f is the the knot on the torus which winds twice around the first S^1 and three times around the second S^1. Let C_f be the mapping cone on f, i.e. the space obtained from $\left(S^1 \times [a, b]\right) \sqcup \left(S^1 \times S^1\right)$ by contracting $\{(x, a) \mid x \in S^1\}$ to a point and identifying (x, b) with $f(x)$, for every $x \in S^1$. Compute:
  (a) $\pi_1 C_f$;
  (b) the cohomology ring $H^*(C_f, \mathbb{Z})$.

(4) If $\{U_i \subset \mathbb{T}^n\}_{1\le i\le k}$ is a good cover of the n-torus $\mathbb{T}^n$ (this means that $\bigcap_{i\in I} U_i$ is either empty or contractible, for every $\emptyset \neq I \subseteq [k]$), prove the inequality $k > n$.

(5) Let G be a finite group acting freely on a connected CW complex K by cellular maps, and let K/G be the quotient. Prove the isomorphism

$$H^*(K, \mathbb{Z}/p) \cong H^*(K/G, \mathbb{Z}/p)$$

if p does not divide $|G|$.