GEOMETRY/TOPOLOGY PRELIMINARY EXAMINATION, JUNE 2025

INSTRUCTIONS:

- There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.
- In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I

Do **three** of the following five problems.

Problem 1. (i) For any vector field X, show that the Lie derivative $\mathcal{L}_X$ on p-forms commutes with the exterior derivative: we have $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$.

(ii) For any pair of vector fields X, Y and any p-form η, show that we have

$$(\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X)(\eta) = \mathcal{L}_{[X,Y]}(\eta).$$

Problem 2. Consider the $2n \times 2n$ matrix J with block decomposition

$$J = \left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right],$$

where I_n is the $n \times n$ identity matrix. Show that

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{ A \in M_{2n\times 2n}(\mathbb{R}) \ : \ A^T J A = J \right\}$$

is an embedded submanifold of the vector space $M_{2n\times 2n}(\mathbb{R})$ of all $2n \times 2n$ real matrices. Compute its dimension and describe its tangent space at the identity as a subspace of $M_{2n\times 2n}(\mathbb{R})$.

Problem 3. (i) State the definition of an involutive distribution on a smooth manifold.

(ii) Suppose that D is an involutive distribution spanned by two vector fields X and Y, and that θ is a 1-form such that $\theta(X) = \theta(Y) = 0$. Show that $d\theta(X, Y) = 0$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}_{>0}$ be a positive smooth function, and consider the surface of revolution

$$M = \left\{ (f(u)\cos(v), f(u)\sin(v), u) \ \middle|\ u \in \mathbb{R}, 0 \le v < 2\pi \right\} \subset \mathbb{R}^3$$

(i) Show that M is a submanifold, and using (u, v) as global coordinates on M write down the metric ι^*g on M induced by pulling back the standard Euclidean metric g on $\mathbb{R}^3$ along the inclusion $\iota : M \to \mathbb{R}^3$.

(ii) Derive the geodesic equations for this metric, and determine for which values of u_0 the curve $\{(u_0, v)\}$ is a geodesic on M when parametrized to have constant speed.

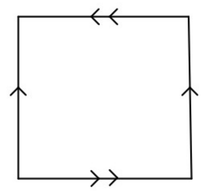
Problem 5. (i) Suppose $F : N \to M$ is a smooth map which is transverse to an embedded submanifold $X \subset M$ of codimension k. Show that $F^{-1}(X)$ is a codimension k embedded submanifold of N.

(ii) Show that if two embedded submanifolds X_1, X_2 intersect transversely in M, then $T_p(X_1 \cap X_2) = T_p(X_1) \cap T_p(X_2)$.

Part II

Do **three** of the following five problems.

Problem 1. Compute the fundamental group of the Klein bottle, i.e. the space obtained by gluing opposite sides of a rectangle as indicated by the arrows in the diagram:



Problem 2.

(1) Give an example of a covering map $f : Y \to X$ between connected spaces such that there exist two points y_1, y_2 such that $f(y_1) = f(y_2) = x$ but the images of the induced maps

$$f_*^{(1)} : \pi_1(Y, y_1) \to \pi_1(X, x), \qquad f_*^{(2)} : \pi_1(Y, y_2) \to \pi_1(X, x)$$

do not agree.

(2) Prove that, for any such example, the images of $f_*^{(1)}$ and $f_*^{(2)}$ must be conjugate inside $\pi_1(X, x)$.

Problem 3. For integers $0 < m \leq n$, consider the natural inclusion $\mathbb{RP}^{m-1} \to \mathbb{RP}^n$ induced by the inclusion $\mathbb{R}^m \subset \mathbb{R}^{n+1}$ as the first m coordinates. We denote the quotient by $\mathbb{RP}^n_m := \mathbb{RP}^n / \mathbb{RP}^{m-1}$, known as a *stunted projective space*.

(1) Compute the fundamental group of $\mathbb{RP}^8_3$.
(2) Compute the homology groups $H_n(\mathbb{RP}^8_3; \mathbb{Z})$ for all $n \geq 0$.
(3) Compute the homology groups $H_n(\mathbb{RP}^8_3; \mathbb{Z}/p)$ for all $n \geq 0$ and primes p.

Problem 4.

(1) Give an example of a connected topological space X with $H_3(X; \mathbb{Z}) = \mathbb{Z}/3$, and $H_n(X; \mathbb{Z}) = 0$ for all $n \neq 0, 3$.
(2) For this space X, compute $H_n(X \times X; \mathbb{Z})$ for all $n \geq 0$.

Problem 5. Consider two small non-intersecting open 2-dimensional disks $D_1, D_2 \subset \mathbb{RP}^2$.

(1) Compute $H_n(\mathbb{RP}^2 \backslash D_1; \mathbb{Z})$ and the map induced on H_n by the inclusion $\mathbb{RP}^2 \backslash D_1 \subset \mathbb{RP}^2$ for all $n \geq 0$.

(2) Compute $H_n(\mathbb{R}\mathrm{P}^2 \backslash (D_1 \coprod D_2); \mathbb{Z})$ for $n \geq 0$.

(3) Let B_1 be the boundary of the closure of D_1 in $\mathbb{R}\mathrm{P}^2$. With respect to your identification in (b), what is the image of the map

$$H_1(B_1; \mathbb{Z}) \to H_1(\mathbb{R}\mathrm{P}^2 \backslash (D_1 \coprod D_2); \mathbb{Z})$$

induced by the inclusion?

Part III

Do **three** of the following five problems.

Problem 1.

(1) Let X be a finite CW complex and let F be any field. Prove that the Euler characteristic $\chi(X)$ is equal to

$$\chi(X) = \sum_n (-1)^n \dim_F \big(\mathrm{H}_n(X, F)\big) \ .$$

(2) Prove that the Euler characteristic of any odd-dimensional closed manifold M is zero.[1]

Problem 2.

Let M be a compact 3-manifold with boundary $i : \partial M \hookrightarrow M$, and consider homology $\mathrm{H}_* = \mathrm{H}_*(-, \mathbb{F}_2)$ with mod-2 coefficients.

(1) Prove equalities of dimensions

$$\dim_{\mathbb{F}_2}\Big(\mathrm{Im}\big((\mathrm{H}_2(M, \partial M) \xrightarrow{\partial} \mathrm{H}_1(\partial M)\big)\Big) = \dim_{\mathbb{F}_2}\Big(\mathrm{Ker}\big((\mathrm{H}_1(\partial M) \xrightarrow{i_*} \mathrm{H}_1(M)\big)\Big)$$

and

$$\dim_{\mathbb{F}_2}\Big(\mathrm{Ker}\big((\mathrm{H}_1(\partial M) \xrightarrow{i_*} \mathrm{H}_1(M)\big)\Big) = \dim_{\mathbb{F}_2}\Big(\mathrm{Coker}\big((\mathrm{H}^1(M) \xrightarrow{i^*} \mathrm{H}^1(\partial M)\big)\Big) \ .$$

(2) Using (a), prove the equality

$$\dim\Big(\mathrm{Im}\big((\mathrm{H}_2(M, \partial M) \xrightarrow{\partial} \mathrm{H}_1(\partial M)\big)\Big) = \frac{1}{2} \dim \mathrm{H}_1(\partial M) \ .$$

[Hint: use the compatibility of Poincaré duality with the long exact sequence of the pair $(M, \partial M)$.]

(3) Conclude that $\mathbb{RP}^2$ is not the boundary of any compact 3-manifold.

Problem 3. Prove the Poincaré Lemma: the de Rham cochains $\Omega^*(\mathbb{R}^n)$ is quasi-isomorphic to $\mathbb{R}[0]$, the cochain complex concentrated in degree zero.

Problem 4.

(1) Let M be a compact odd-dimensional manifold, possibly with boundary. Prove that the Euler characteristic of the boundary is twice that of M:

$$\chi(\partial M) = 2 \cdot \chi(M) \ .$$

(2) Using (1), for *infinitely many* dimensions n, give an example of a closed orientable n-manifold N which is not the boundary of any compact $(n+1)$-manifold.

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[1] You may assume that M is homotopy equivalent to a finite CW complex.

Problem 5. Let M be an n-manifold with orientation μ, i.e., a consistent choice of generator $\mu_x \in \mathrm{H}_n(M, M \setminus x; \mathbb{Z})$ for each point $x \in M$. Recall that a map $f : M \to M$ is *orientation-reversing* if $f_* \mu_x = -\mu_{f(x)}$ for all x. Prove there is an orientation-reversing homeomorphism

$$f : \mathbb{CP}^n \to \mathbb{CP}^n$$

if and only if n is odd.