Algebra Preliminary Examination

September, 2015

Do all of the following questions.

Question 1. Let $f \in \mathbb{Z}[x]$ be a polynomial with integer coefficients. Prove that if f factors nontrivially over $\mathbb{Q}$ then it factors nontrivially over $\mathbb{Z}$.

Question 2. The quaternion group Q has 8 elements, $\{\pm 1, \pm i, \pm j, \pm k\}$, with group structure

$$i^2 = j^2 = k^2 = -1$$
$$ij = k = -ji$$
$$jk = i = -kj$$
$$ki = j = -ik \ .$$

Calculate the character table of Q. Explicitly construct the irreducible representations of the quaternion group Q. Prove your answers are correct.

Question 3. Recall the elementary symmetric polynomials in n variables:

$$e_1 = x_1 + \ldots + x_n \ ,$$
$$e_k = \sum_{1 \le i_1 < \ldots < i_k \le n} x_{i_1} \ldots x_{i_k} \ ,$$
$$e_n = x_1 x_2 \ldots x_n$$

Prove that the elementary symmetric polynomials generate the ring

$$\mathbb{Z}\big[x_1, \ldots, x_n\big]^{\Sigma_n}$$

of invariants of the symmetric group action.

Question 4. State, without proof, the Galois group of $\mathbb{F}_8$ over $\mathbb{F}_2$. How many monic irreducible factors does

$$X^{255} - 1 \in \mathbb{F}_2[x]$$

have, and what are their degrees? Prove your answer.

Question 5. Prove the following form of Hensel's Lemma. Let A be a complete local ring with residue field $\Bbbk$, and let $f \in A[x]$ be a polynomial which reduces to $f_0 \in \Bbbk[x]$. Given $a \in \Bbbk$ for which

$$f_0(a) = 0 \ , \text{ and}$$
$$f_0'(a) \neq 0$$

then there exists a unique element $\alpha \in A$ which is a lift of a and such that $f(\alpha) = 0$.

Question 6. Consider the commutative ring

$$A := \mathbb{C}[x, y]/(y^2 - (x-1)^3 - (x-1)^2)$$

1. Draw $\mathsf{Spec}(A)$.
2. Calculate the integral closure of A.
3. Calculate the integral closure of $\mathbb{Z}[2i]$, and prove that your answer is correct.
4. Consider the maximal ideals of

$$B = \mathbb{C}[x, y]/(x^4 + y^4 - x^2 y^2)$$

according to the rank of their Zariski cotangent spaces; prove your answer.