Algebra Preliminary Exam
July 16, 2014
Answer All Questions

(1) Show that there do not exist two irreducible polynomials $a(x)$ and $b(x)$ in $\mathbf{Q}[x]$ of of degrees 6 and 7 respectively which have isomorphic splitting fields.
(2) Let F/E be a finite extension of fields of degree $d > 1$. Prove or disprove: F is not isomorphic (as a field) to E.
(3) Prove or disprove: every finite group G has a faithful, irreducible representation over the complex numbers.
(4) Let G be a finite group. Let V be a faithful representation of G of dimension d over the complex numbers. Suppose that $|\chi_V(g)| = d$, where χ is the character of V. Prove that g is central.
(5) Let R be a commutative ring, and let $A = R[[T]]$. Prove that A is a local ring if and only if R is a local ring.
(6) Let R be a commutative local ring, and let $x \in R$ be an element which is not a unit. Suppose that for all exact sequences:

$$0 \to A \to B \to C \to 0$$

the following sequence is also exact:

$$0 \to A/xA \to B/xB \to C/xC \to 0.$$

Prove or disprove: $x = 0$ in R.