Algebra Preliminary Exam, June 2011

1. (a) Show that any group of order 42 contains a normal subgroup of index 6.

   (b) Find (with proof) a group of order 42 that contains a subgroup H such that $H \cong S_3$ and such that H is not normal.

2. Let k be a field.

   (a) Prove that any semi-simple k-algebra of dimension ≤ 3 is commutative.

   (b) Does the result of (ii) remain true if you omit the hypothesis of semi-simplicity?

3. Let G be a finite group and let ρ be an representation of G on an n-dimensional vector space V over the field $\mathbb{C}$ of complex numbers. Suppose that for every element $g \in G$, there exists a basis for V with respect to which the linear automorphism $\rho(g)$ has the form

$$\begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

   (i.e. 0s below the diagonal, 1s on the diagonal, and unspecified entries above the diagonal). Prove that ρ is trivial, i.e. that $\rho(g)$ acts as the identity on V for every $g \in G$.

4. Find the Galois group of $x^4 + 1$ over each of the following fields: $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{F}_3$, $\mathbb{F}_5$.

5. Let A be a commutative ring with 1, and let M be a finitely generated A-module.

   (a) If $\mathfrak{m}$ is a maximal ideal of A, prove that $M/\mathfrak{m}M$ is non-zero if and only if the localization $M_{\mathfrak{m}}$ is non-zero.

   (b) Is the analogous statement true if we replace $\mathfrak{m}$ by a non-maximal prime ideal of A? Carefully explain why or why not.

6. Suppose that A is a commutative ring with 1.

   (a) If $N \subset M$ are A-modules and $N_{\mathfrak{m}} = M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$, show that $N = M$.

   (b) Suppose that A has only finitely many maximal ideals. If $A_{\mathfrak{m}}$ is Noetherian for all maximal ideals $\mathfrak{m}$, show that A is Noetherian.

7. Let $A = \mathbb{C}[x, y, z]/(x^2 + y^2 - 2z^2)$, and let I be the ideal of A generated by $x - y$.

   (a) Prove that I is a radical ideal.

   (b) Find all the minimal primes of I.

   (c) Determine the height of each of the prime ideals you found in (b).