Algebra Preliminary Examination

June 2012

Answer all questions. This exam has seven problems on two pages.

1. Suppose K/k is an algebraic extension and let $G = Aut_k(K)$.

i.) Prove that K/k is Galois if and only if for every $\alpha \in K$, the minimal polynomial of α over k is equal to

$$\prod_{g \in G/G_\alpha} (X - g(\alpha))$$

where G_α is the isotropy group of α (i.e., those elements of G that fix α).

ii.) Give an example of K/k such that G_α is nontrivial for every $\alpha \in K$.

2. Let k be a field and $f \in k[X]$ a monic irreducible polynomial of degree n. Let K denote a splitting field of f.

i.) Prove that $[K : k]$ divides $n!$.

ii.) If $k = \mathbb{F}_p$, prove that $[K : k] \leq n$.

3. Suppose p_1, p_2, and p_3 are primes (not necessarily distinct). Prove that any group of order $p_1 p_2 p_3$ is solvable.

4. Let A be a commutative local ring and M and N two finitely generated A-modules. Show that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

5. Let A be an integrally closed integral domain, K its field of fractions and L a finite normal separable extension of K. Let G be the Galois group of L over K and let B the integral closure of A in L. Show that $\sigma(B) = B$ for all $\sigma \in G$, and that

$$A = B^G = \{ \ x \in B \mid \sigma(x) = x \text{ for all } \sigma \in G \ \}.$$

There are two more problems overleaf.

6. Prove the Hilbert Basis Theorem: if A is a Noetherian commutative ring, then the polynomial ring $A[X]$ is Noetherian.

7. In this problem k is a field, G is a finite group and $k[G]$ is the group ring.

i.) Suppose k is an algebraically closed field and of characteristic zero. Let $G = C_n$ be the cyclic group of order n. Show that every finitely generated $k[C_n]$-module is a direct sum of modules which are of dimension 1 over k.

ii.) Show (i) is false if we drop the assumption that k be of characteristic zero.

iii.) Show (i) is false if we drop the assumption that k be algebraically closed.

iv.) Show (i) is false if we replace C_n by the dihedral group of order 8.