Algegra Preliminary Exam: September 2000

1. Let F be a field of characteristic 0 and consider the field $F(x)$, where x is transcendental over F (i.e., satisfies no polynomial equation with coefficients in F). Let $G \subset Aut(F(x))$ be the group of automorphisms geneterated by the automorphism over F sending x to $x+1$.
a. Determine $F(x)^G$.
b. Determine the Galois group $Gal(F(x)/F(x)^G)$.

2. Let k be a field and let A be a finitely generated commutative k-algebra. Show that A is Artinian if and only if it is finite dimensional as a k-vector space.

3. Let $\mathbb{Q}(\zeta_{18})$ be the cyclotomic field obtained by adjoining to $\mathbb{Q}$ the roots of $x^{18}-1$.
a. Determine $Gal(\mathbb{Q}(\zeta_{18})/\mathbb{Q})$.
b Describe all fields F, $\mathbb{Q} \subset F \subset \mathbb{Q}(\zeta_{18})$.

4. Let R be a ring and $\mathfrak{m} \subset R$ a maximal ideal.
a. Show that $R_{\mathfrak{m}}$ is a local ring.
b. Show that $R = \cap_{\mathfrak{m}} R_{\mathfrak{m}}$ whenever R is an integral domain, where the intersection is indexed by all maximal ideals of R.

5 Let G be given by a set X of generators and a set R of relations (so that G equals the quotient of the free group on X by the normal subgroup generated by R), and similarly let G' be given by a set X' of generators and a set R' of relations.
a. Give generators and relations for a group $G * G'$ which satisfies

$$Hom_{grps}(G * G', K) = Hom_{grps}(G, K) \times Hom_{grps}(G', K)$$

for all groups K.
b. Prove that the property given in (a.) determines $G * G'$ up to isomorphism.

6. Find the injective envelope for the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$, $n \geq 0$.