NU Math Department ALGEBRA PRELIM September 18, 2002

These questions vary in difficulty. Budget your time so that your work reflects as much of your knowledge as possible. In working these problems, you may quote standard facts and theorems so long as they are not essentially equivalent to what you are trying to prove. **Good luck!**

1. Let C be a cyclic group of order n. Show that for each $d \mid n$, C contains a unique subgroup of order d .

2. Let G be a group of order pq, where p and q are primes with $p < q$.

   (a) Show that G has a normal cyclic subgroup $Q = \langle x \rangle$ of order q.

   (b) Show that G has a cyclic subgroup $P = \langle y \rangle$ of order p, and that $yxy^{-1} = x^i$ for some $i \in \{1, 2, \ldots, q-1\}$.

   (c) Show that $G \cong P \ltimes Q$ (the semidirect product).

   (d) Explain why the semidirect product is in fact the direct product if p does not divide $q-1$.

3. Let $f(X) = X^4 + 2$ and let $i = e^{2\pi i/4}$.

   (a) Find a splitting field $K \subset \mathbb{C}$ for $f(X)$ **over** $\underline{\mathbb{Q}(i)}$. Compute $G(K/\mathbb{Q}(i))$. Determine all subgroups of $G(K/\mathbb{Q}(i))$ and the corresponding lattice of subfields of K. Identify which are normal and which are not.

   (b) Do the same as in (3a), but this time for K a splitting field of $f(X)$ over $\mathbb{Q}$.

   (c) Determine the Galois group of the splitting field of $f(X)$ (but not the subgroups, subfields, etc.) over $\mathbb{Q}(\sqrt{2})$.

   (d) Find the Galois group of the splitting field of $f(X)$ over $\mathbb{F}_3$.

4. Prove that a free module over a domain is torsion free. (You may assume the module is finitely generated if you wish.) Then give a counterexample to show why the hypothesis "domain" is necessary.

5. Let $\gcd(n, m) = 1$, and denote by ζ_n (respectively ζ_m) a primitive n^{th} (respectively m^{th}) root of unity, both elements of $\mathbb{C}$.

   (a) Prove that the compositum $\mathbb{Q}(\zeta_n)\mathbb{Q}(\zeta_m)$ contains a primitive nm^{th} root of unity, and deduce that $\mathbb{Q}(\zeta_{nm}) = \mathbb{Q}(\zeta_n)\mathbb{Q}(\zeta_m)$.

   (b) Show that $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$.

   (c) Give a counterexample to show that these assertions are not necessarily true if $\gcd(n, m) \neq 1$.

6. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{3})$ coincides with $\mathbb{Z}[\sqrt{3}]$.

7. Let A be a left Noetherian ring. Show that every element $a \in A$ which is left invertible is actually two-sided invertible (Hint: Use the fact that every surjective endomorphism of a Noetherian module is an isomorphism).

8. An element x of a (not necessarily commutative) ring R is said to be strongly nilpotent if there exists an integer $n > 0$ such that every product of elements of R in which at least n factors coincide with x is zero.

   (a) Show that the set J of strongly nilpotent elements is a two-sided ideal in R.

   (b) Show that J is contained in the Jacobson radical of R.

   (c) Show that if R is Artinian then J coincides with the Jacobson radical of R.