Math 470 Algebra

September 2003

1.a) Let C_p be the cyclic group of order p and let X be a finite set so that number of elements of X is prime to p. Show that if C_p acts on X, then this action must have a fixed point.

b) Now prove Cauchy's Theorem: if p divides the order of a finite group, then G has an element of order p. To get started you might notice that the set

$$Y = \{\ (x_1, x_2, \dots, x_p) \mid x_1 x_2 \cdots x_p = 1\ \} \subset G^p$$

contains the element $(e, e, \dots, e)$ where e in the identity element of G.

2. Find the splitting field F of the polynomial $x^4 - 2$ over the rational numbers $\mathbb{Q}$. Then identify the Galois group of F over $\mathbb{Q}$.

3. a)Let A be a commutative local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated A-module. Prove that if $M/\mathfrak{m}M = 0$, then $M = 0$.

b) Now show that if $f : M \to M$ is an A-module endomorphism so that $\bar{f} : M/\mathfrak{m}M \to M/\mathfrak{m}M$ is onto, then f is onto.

4. Recall that two $n \times n$ matrices A and B over a field are similar if there is an invertible $n \times n$ matrix Q so that

$$A = QBQ^{-1}.$$

A partition of a positive integer n is a sequence of positive integers $n_1 \geq n_2 \geq \ldots \geq n_k$ so that $n_1 + \cdots + n_k = n$. Let $P(n)$ be the number of distinct partitions of n. For example, $P(4) = 5$.

Prove that up to similarity of matrices, there are exactly $P(n)$ $n \times n$ matrices A so that $A^n = 0$ (same n).

5. Suppose there is a commutative diagram of abelian groups

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 \end{array}$$

in which the rows are exact. Prove that if f_1 and f_3 onto, and f_4 one-to-one, then f_2 is onto.

6. Let C_3 be the cyclic group of order 3 with generator σ. Define a subspace V of $\mathbb{C}^3$ by

$$V = \{\ (x, y, z) \mid x + y + z = 0\ \}.$$

Then C_3 acts linearly on V by $\sigma(x, y, z) = (z, x, y)$. Write V as a direct sum of simple modules over the group ring $\mathbb{C}[C_3]$.