Algebra Preliminary Exam

September 22 2003

1. Exhibit a 3-Sylow subgroup in the symmetric group S_6 and describe its structure. Find the normalizer of this subgroup and compute the total number of Sylow 3-subgroups in S_6.

2. Show that every group of order 15 is abelian.

3. Let $f = X^3 + pX + q \in k[X]$ be an irreducible polynomial of degree three over a field k of characteristic zero. Denote by x_1, x_2, x_3 the roots of f in the algebraic closure of k.
   a) Show that $\Delta = [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]^2$ is an element of k.
   b) Show that the Galois group of f is either S_3 or A_3 depending on whether Δ is a square in k or not.

4. Let F be a finite field with 27 elements
   a) Describe the structure of the additive group of F.
   b) Describe the structure of the multiplicative group of F.
   c) Describe the structure of the group of field automorphisms of F.

5. Show that the factorring $\mathbb{Z}[X]/(X^2 - X + 1)$ is an integrally closed domain, describe its field of fractions.

6. An element x of a (not necessarily commutative) ring R is said to be strongly nilpotent if there exists an integer $n > 0$ such that every product of elements of R in which at least n factors coincide with x is zero.
   a) Show that the set J of strongly nilpotent elements is a two-sided ideal in R.
   b) Show that J is contained in the Jacobson radical of R.
   c) Show that if R is Artinian then J coincides with the Jacobson radical of R.