Algebra Preliminary Exam
September 2009

Do all of the following questions.

1. Let G be a group, and V and W be two irreducible representation of G over a field k.

(a) If V and W are not isomorphic, show that the G-subrepresentations of the product $V \times W$ are precisely those on the following list:

$$\{0\}, \quad V \times \{0\}, \quad \{0\} \times W, \quad V \times W.$$

(b) Is the same result true if $V \cong W$?

2. Let G be a group of order 12.

(a) Describe the possible structures for the group ring $\mathbb{C}[G]$ as a product of simple algebras over $\mathbb{C}$.

(b) How many irreducible representations can G have, and what are their dimensions? (There may be different possibilities here, depending on the structure of G.)

3. (a) Find the Galois group of the splitting field K of $x^3 + 3$ over $\mathbb{Q}$ and describe how that group acts on the roots of this polynomial.

(b) Find all intermediate subfields between $\mathbb{Q}$ and K.

4. Which of the following rings are Noetherian? (In each case, justify your answer.)

(a) The subring of $\mathbb{C}[x, y]$ consisting of all polynomials $f(x, y)$ such that $f(x, 0)$ is a constant polynomial.

(b) The subring of $\mathbb{C}[x, y]$ consisting of all polynomials $f(x, y)$ whose gradient vanishes at the point $x = y = 0$.

5. Consider the following modules over the ring of integers $\mathbb{Z}$:

$$M = \bigoplus_p \mathbb{Z}/p\mathbb{Z} \qquad \text{and} \qquad N = \prod_p \mathbb{Z}/p\mathbb{Z},$$

where both the direct sum and the product are taken over the set of all prime numbers.

(a) Show that the localizations $M_{\mathfrak{P}}$ are finitely generated $\mathbb{Z}_{\mathfrak{P}}$-modules for any prime ideal $\mathfrak{P}$ of $\mathbb{Z}$.

(b) Is (a) true with M replaced by N?

6. Let M be a torsion $\mathbb{Z}$-module. Show that $M \otimes \mathbb{Q}/\mathbb{Z} = 0$.