Algebra Preliminary Exam

September 2011

(1) Let $\mathbf{F}$ be a field of cardinality 729. For the following question only, you do not need to justify your answer.
   (a) What is the additive group $\mathbf{F}$ as an abstract abelian group?
   (b) What is the multiplicative group $\mathbf{F}^\times$ of units of $\mathbf{F}$ as an abstract abelian group?
   (c) What is the group of field automorphisms of $\mathbf{F}$?

(2) Prove that $\mathbf{Z}[x]$ is a unique factorization domain.

(3) Let K be a field, and let $R = K[x,y]/(y^2 - x^3)$. Prove that the localization of R at $\mathfrak{m} = (x, y)$ is not a discrete valuation ring.

(4) Let G be a finite group, and let V be a characteristic zero representation of G. Suppose that for every $g \in G$, the fixed space $V^g \subset V$ of $v \in V$ such that $gv = v$ has dimension at least $\frac{1}{2}\dim(V)$. Prove that there exists a $v \in V$ such that $gv = v$ for *all* $g \in G$.

(5) Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be local noetherian rings. Suppose that $\phi : A \to B$ is a map such that $\phi(\mathfrak{m}) \subset \mathfrak{n}$, and suppose that:
   (a) $A/\mathfrak{m} \to B/\mathfrak{n}$ is an isomorphism,
   (b) $\mathfrak{m} \to \mathfrak{n}/\mathfrak{n}^2$ is surjective,
   (c) B is finitely generated as an A-module.
   Prove that ϕ is surjective.

(6) Let p be prime. Prove that $x^p - x - 1$ is irreducible over $\mathbf{F}_p$. What is the Galois group of its splitting field?