Algebra Preliminary Exam

September 12 1997

1. Show that each group of order 15 is abelian.

2. Show that for each n the symmetric group S_n may be generated by two elements: the transposition $\sigma = (1, 2)$ and the long cycle $\tau = (1, 2, ..., n)$.

3. Find the Galois group of the polynomial $x^8 - 1$ over $\mathbb{Q}$. Describe the splitting field K of this polynomial: give the degree of K over $\mathbb{Q}$, indicate the minimal polynomial of ξ_8 over $\mathbb{Q}$, find all intermideate fields between K and $\mathbb{Q}$.

4. Let F be a finite field with 27 elements
   a) Describe the structure of the additive group of F.
   b) Describe the structure of the multiplicative group of F.
   c) Describe the structure of the group of field automorphisms of F.

5. Show that the ring $\mathbb{Z}[X]/(X^2 + X + 1)$ is an integrally closed domain, describe its field of fractions.

6. Let R be an Artinian ring and let M be a finitely generated R-module. Show that an injective endomorphism of M is an automorphism.