Algebra preliminary examination, Fall 1991.

1. (a) Define semisimple rings and give an example of such a ring.
   (b) Explain why $\mathbf{Z}$ is not semisimple.
   (c) List all non-isomorhic semisimple rings with 81 elements.

2. Give at least 2 equivalent characterizations of the Jacobson radical of a ring R. Then compute the Jacobson radicals of the following rings:

$$\mathbf{Z}/8\mathbf{Z}, \quad \mathbf{Z}/60\mathbf{Z}, \quad \mathbf{Q}[x]/(x^3 - 5x).$$

3. Define what does it mean for a module M to be artinian. Then prove:

Lemma. *If M is an artinian module and $f : M \to M$ is an injective homomorphism, then f is an isomorphism.*

4. (a) Define projective module.
   (b) Define local ring.
   (c) Prove that a finitely generated projective module over a local ring is free.

5. Find the splitting field over $\mathbf{Q}$ of the polynomial $x^4 - 1$. Compute its galois group. Describe its subgroups and corresponding subfields and indicate which subfields are normal.

6. Let F_n be a field with n elements. List all the subfields of F_{16}, F_{32}, F_{64}.

7. Let G be a finite group and p be the smallest prime dividing $|G|$. Prove that any subgroup of index p is normal in G.

8. (a) Define simple group.
   (b) Define the alternating group A_n and identify it with some more elementary groups for $n = 2, 3, 4$.
   (c) Prove that the only simple group of order 60 is A_n