Algebra preliminary Examination, Fall 1996

1. Let G be a group of order 63.

   (a) Show that every 7-Sylow subgroup G_7 of G is normal.

   (b) let G_3 be a 3-Sylow subgroup of G. Show that the canonical map $G_3 \hookrightarrow G \to G/G_7$ is an isomorphism. Conclude that G is a semidirect product of G_3 and G_7.

   (c) List all possible groups of order 63 up to isomorphism.

2. Show that for every $n > 0$ the symmetric group S_n can be generated by two elements (exhibit them).

3. Find the Galois group of the polynomial $x^{10} - 1$ over the field $\mathbf{Q}$ of rational numbers. Describe the splitting field K of this polynomial: gibe the degree $[K : \mathbf{Q}]$, find the minimal polynomial of the primitive root of $x^{10} - 1 = 0$ over $\mathbf{Q}$ and find all the intermediate fields between K and $\mathbf{Q}$.

5. Show that the ring $\mathbf{Z}[x]/(x^2 + 1)$ is an integrally closed domain.

6. Let F be a field and $A \subset \mathrm{Mat}_2(F)$ be the subring consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Find the Jacobson radical of A and all the simple A-modules.