PRELIMINARY EXAM IN ALGEBRA SPRING 2022

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then we will count the three problems with highest scores.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Group Theory, Homological Algebra, and PIDs

Do **three** of the following five problems.

(1) Let $n = 2p$ where p is an odd prime. Find, up to isomorphism, all groups of order n.

(2) Let p be a prime number.
  (a) Let G be a finite group and $P \subseteq G$ a p-Sylow subgroup. Show that if $H \subseteq G$ is a subgroup, then H has p-Sylow subgroup. You might consider an action of H on G/P.
  (b) Let H be finite group. Show H is a subgroup of $\mathrm{Gl}_n(\mathbb{Z}/p)$ for some n.
  (c) Conclude every finite group has a p-Sylow subgroup. Note the order of $\mathrm{Gl}_n(\mathbb{Z}/p)$ is $p^k m$ where m is prime to p and $k = \binom{n}{2} = n(n-1)/2$.

(3) Let $\mathbb{Z}[C_2]$ be group ring over the integers of the cyclic group $C_2 = \{1, \tau\}$ with $\tau^2 = 1$.
  (a) Let C_2 act on $\mathbb{Z}$ by $\tau(n) = n$. Write down a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}[C_2]$-module.
  (b) Let $\mathbb{Z}(\mathrm{sgn})$ be the $\mathbb{Z}[C_2]$-module with $\tau(n) = -n$. Calculate

$$Tor_n^{\mathbb{Z}[C_2]}(\mathbb{Z}, \mathbb{Z}(\mathrm{sgn})), \qquad n > 0.$$

(4) Let R be a commutative integral domain. Recall that an R-module M is divisible if for all $a \in R$ and all $x \in M$, there is an element $y \in M$ so that $ay = x$.
  (a) Show that any injective R-module is divisible.
  (b) Let R be a principle ideal domain. Show any divisible module is injective.

(5) Let R be a principal ideal domain. If M is an R-module and $a \in R$, let $M(a) \subseteq M$ be the submodule of a-torsion, so $x \in M(a)$ if $a^n x = 0$ for some n. Also M is a torsion module if for all $x \in M$, $ax = 0$ for some $0 \neq a \in R$.
  (a) Show that a and b are relatively prime, then $M(a) \cap M(b) = 0$.
  (b) Show that if M is finitely generated and torsion there are distinct primes $p_i \in R$ so that

$$M(p_1) \oplus \cdots \oplus M(p_n) = M.$$

(c) Now let V be a finite vector space over an algebraically closed field and let $T : V \to V$ be a linear transformation. Show T has an eigenvalue.

Part II. Galois Theory and Representation Theory

Do **three** of the following five problems.

(1) Suppose that $p_1, \ldots, p_n, q_1, \ldots, q_n$ are distinct primes. Show that
$$\mathbb{Q}\left(\sqrt[p_1]{20}, \ldots, \sqrt[p_n]{20}\right) \cap \mathbb{Q}\left(\sqrt[q_1]{22}, \ldots, \sqrt[q_n]{22}\right) = \mathbb{Q}.$$

(2) Let $a, b \in \mathbb{Z}$. Assume that $K := \mathbb{Q}[\sqrt{a}, \sqrt{b}]$ is a degree-4 extension of $\mathbb{Q}$. Find three distinct proper intermediate fields $\mathbb{Q} \subsetneqq F \subsetneqq K$ and prove that there are no others.

(3) Let $K/\mathbb{Q}$ be a Galois extension of degree 2^{2022}. Show that there is non-square $d \in \mathbb{Z}$ such that $\sqrt{d} \in K$.

(4) Let G be a finite group, and let $g \in G$. Show that g is conjugate to g^{-1} if and only if $\chi(g) \in \mathbb{R}$, for every character χ of G.

(5) Let p be a prime number. Find the degree of the extension $\mathbb{Q}\left(\cos\left(\frac{2\pi}{p}\right)\right)/\mathbb{Q}$.

Part III. Commutative Algebra

Do **three** of the following five problems.

(1) For the ring $R = \mathbb{C}[x, y, z]/(xy - z^2)$, construct an inclusion of a polynomial ring A over $\mathbb{C}$ such that the map $A \to R$ is finite. Compute the Krull dimension of R.

(2) Let A be a finite abelian group. Prove that the group ring $\mathbb{C}[A]$ is isomorphic as a ring to a product of copies of $\mathbb{C}$.

(3) Let $\mathfrak{m} \in \mathsf{Spec}(R)$ be a maximal ideal of a finitely-generated $\mathbb{C}$-algebra $R = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_d)$. Prove that the rank of the Jacobian matrix $(\partial f_i/\partial x_j)(\mathfrak{m})$ is $n - d$ if and only if $R_{\mathfrak{m}}$ is a regular local ring.

(4) How many solutions does $x^2 + 4x + 2 = 0$ in $\mathbb{Z}_7$, the 7-adic integers? Prove your answer.

(5) Let R be a commutative ring, and M be a finitely generated R-module. Let $I \subset R$ be an ideal such that $IM = M$. Prove that there exists an element $a \in 1 + I$ such that $aM = 0$.