Algebra Preliminary Examination

Northwestern University, June 2017

Do all of the following questions. Each question is worth 0.5 points.

Question 1. Compute the Krull dimension of $\mathbb{Z}[X,Y]/\langle XY-1\rangle$.

Question 2. Let $Q = \{\pm 1, \pm x, \pm y, \pm z\}$ be the quaternion group, with $x^2 = y^2 = z^2 = -1$ and $xy = -yx$, $xz = -zx$, $yz = -zy$.

1. Show that there is an irreducible representation ρ of Q on $\mathbb{C}^{\oplus 2}$ given by matrices

$$\rho(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \rho(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(z) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

2. Find all other irreducible representations of Q over $\mathbb{C}$. Justify your answer.

Question 3. Consider the power series ring $\mathbb{C}[[T]]$ of one variable. Let $\mathbb{C}((T))$ be its fraction field.

1. Show that for every integer $n \geq 1$, $\mathbb{C}((T^{1/n})) := \mathbb{C}((T))[X]/\langle X^n - T\rangle$ is a cyclic field extension of $\mathbb{C}((T))$ of degree n.
2. Show that $\mathbb{C}((T^{1/n}))$ is a splitting field of the polynomial $X^n - (T + T^2)$ over $\mathbb{C}((T))$.

Question 4. Let R be a ring. Recall that the *Jacobson radical* of R is the left ideal N that is the intersection of all maximal left ideals of R.

1. Show that N is a two-sided ideal. (This was proved in class, but please write down a proof.)
2. Show that N is also the intersection of all maximal right ideals of R.

Question 5 Let G be a finite group, and $Z \subset G$ its center. Let V be an irreducible representation of G over $\mathbb{C}$.

1. Show that for every element $z \in Z$, z acts on V by a scalar.
2. Show that

$$(\dim_{\mathbb{C}} V)^2 \leq \frac{|G|}{|Z|}.$$

Question 6 Let R be a (commutative) integral domain. We say that an R-module M is *torsion-free* if for every nonzero element $r \in R$, the endomorphism $m \mapsto rm$ of M is injective.

1. Let $I \subset R$ be a principal ideal. Prove that $I \otimes_R I$ is a torsion-free R-module.
2. Exhibit an integral domain R and an ideal I such that $I \otimes_R I$ is not a torsion-free R-module.