Algebra Preliminary exam

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such. For multi-part questions, you may use the results of earlier parts in later steps without penalty, regardless of whether you completed the earlier proof.

Part 1

1. Consider the functor $F : \mathrm{Grp} \to \mathrm{Grp}$ given by $G \mapsto G/[G, G]$, the abelianization of G.
   (a) Show that every natural transformation $T : F \implies F$ is determined by its value on $\mathbb{Z}$.
   (b) Describe all natural transformation $T : F \implies F$, and all natural automorphisms of F.
2. Let V be a finite-dimensional vector space over a field k and let $q : V \times V \to k$ be a non-degenerate symmetric bilinear form.
   (a) Suppose $W \subset V$ is a subspace such that $q(x, y) = 0$ for all $x, y \in W$. Show that $\dim_k W \leq \lfloor \dim_k V/2 \rfloor$.
   (b) Assume that $k = \mathbb{C}$. Show that there exists a subspace W with $\dim_k W = \lfloor \dim_k V/2 \rfloor$ such that $q(x, y) = 0$ for all $x, y \in W$.
   (c) Give an example to show that the conclusion of the previous part is false when $k = \mathbb{R}$.
3. Find the possible conjugacy classes of a matrix A with characteristic polynomial $(x^2 + x + 1)(x - 1)^3$ over
   (a) $\mathbb{Q}$;
   (b) $\mathbb{F}_3$.
4. Let V be a finite-dimensional vector space over a field k. Consider the map
$$T : \Lambda^2(V) \otimes V \to \Lambda^3(V)$$
   defined by $(v \wedge w) \otimes u \mapsto v \wedge w \wedge u$.
   (a) Show that T is surjective.
   (b) Compute $\dim \ker(T)$ in terms of $\dim V$.
   (c) Show that $\ker(T)$ is generated by elements of the form $(u \wedge v) \otimes w - (v \wedge w) \otimes u$. (Hint: Count dimensions, and note that for 3 distinct vectors u, v, w, T sends $(u \wedge v) \otimes w$, $(v \wedge w) \otimes u$, $(w \wedge u) \otimes v$ to the same vector $u \wedge v \wedge w$.)
5. Let $R = k[x]/x^3$ where k is a field, and let M be the R-module R/x^2.
   (a) Construct a free resolution of M.
   (b) Compute $\mathrm{Ext}^i_R(M, M)$ and $\mathrm{Tor}^R_i(M, M)$.

Part 2

1. (a) Let F be a field of characteristic p. Show that if $X^p - X - a$ is reducible in $F[X]$, then it splits into distinct linear factors in $F[X]$.
   (b) For every prime p, show that $X^p - X - 1$ is irreducible in $\mathbb{Q}[X]$.

2. Let $f \in \mathbb{F}_p[X]$ be a polynomial of degree
$$n = \prod_{i=1}^{k} p_i^{r_i},$$
where the p_i are distinct primes. Show that f is irreducible over $\mathbb{F}_p$ if and only if
   (a) $\gcd(f(X), X^{p^{n/p_i}} - X) = 1$ for all $1 \leq i \leq k$, and
   (b) $f(X)$ divides $X^{p^n} - X$.

3. Let F be a field and let G be a group. Show that every finite set $\{\chi_1, \dots, \chi_m\}$ of distinct group homomorphisms $G \to F^\times$ is linearly independent over F. In other words, show that if $a_1, \dots, a_m \in F$ and
$$a_1\chi_1 + \cdots + a_m\chi_m = 0$$
as a function $G \to F$, then
$$a_1 = \cdots = a_m = 0.$$

4. Prove that every finite separable field extension is simple.

5. (a) Let G be a finite group, and let (V, π) be a finite-dimensional representation of G over a field F of characteristic 0. Prove Maschke's theorem: (V, π) is completely reducible.
   (b) Classify all finite-dimensional representations of the symmetric group S_3 over $\mathbb{C}$. (In other words, determine the irreducible representations of S_3.)

Part 3

1. Let $K = \mathbb{Q}[\sqrt{11}]$. Let $x \mapsto \bar{x}$ denote the non-trivial Galois automorphism of K.
   (a) Let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. Prove that $\mathcal{O}_K = \mathbb{Z}[\sqrt{11}]$.
   (b) Prove that the ideal $\sqrt{11}\,\mathcal{O}_K$ is a prime ideal.
   (c) Let $11 \neq p \in \mathbb{Z}$ be an odd prime number. Prove that the ideal $p\,\mathcal{O}_K$ is either prime, or factors as $\mathfrak{p}\bar{\mathfrak{p}}$ where $\mathfrak{p} \subset \mathcal{O}_K$ are prime ideals. Here, $\bar{\mathfrak{p}} = \{\bar{x} : x \in \mathfrak{p}\}$.

2. Let R be a finitely generated $\mathbb{C}$-algebra, and let $\mathfrak{n} \subset R$ be the nilradical. Prove that there is some positive integer $m \geq 1$ such that $x^m = 0$ for every x in $\mathfrak{n}$.

3. Let $R = \mathbb{C}[x, y, z]/(y^2 + 2yx - xz)$
   (a) Prove that R is an integral domain.
   (b) Express R as an integral extension of a polynomial ring with complex coefficients, thereby verifying Noether Normalization in this instance.
   (c) Determine the Krull dimension of R.

4. Throughout, let K be a field.
   (a) Let $A \subset K$ a ring, and suppose that K is an integral A-algebra. Prove that A is a field.
   (b) Let L be a finitely generated K-algebra that is a field. Prove Zariski's lemma: L is a finite extension of K.
   (c) Let R and S be finitely generated K-algebras, and let $f : R \to S$ be a map of K-algebras. Prove that the inverse image of a maximal ideal is maximal.

5. Let R be a noetherian ring, and let M be a finitely generated R-module.
   (a) Suppose that $p \subset R$ is a prime ideal such that the localization M_p is zero. Then, show that there is some element $f \in R \setminus p$ such that $M[1/f]$ is zero.
   (b) Define the support of M to be the set $\mathfrak{q} \in \operatorname{Spec} R$, such that $M_{\mathfrak{q}} \neq 0$. Prove that the support of M is closed.