Preliminary Exam in Algebra Fall 2018

Solve all of the following problems. Each question is worth 1/2 point.

(1) Suppose that $R \subset S \subset T$ are commutative rings, that T is integral over S, and that S is integral over R. Is it true that T is integral over R? Prove or give a counterexample.

(2) Fix a positive integer n. Let $\mathcal{F}$ be the functor from abelian groups to abelian groups taking a group G to G/nG. Compute the left derived functors $L_i\mathcal{F}(G)$ for every finitely generated abelian group G.

(3) Suppose G is a finite group that acts on a finite set X. For $g \in G$, let X^g be the set of fixed points of g in X. For $k \in \{1, 2\}$ show that the action is k-transitive iff

$$\frac{1}{|G|} \sum_{g \in G} |X^g|^k = k.$$

(4) Suppose that $F \subset K$ is a Galois extension, and that $F \subset L_1, \ldots, L_n \subset K$ are intermediate Galois extensions. Identify $\mathrm{Gal}(K/L_i)$ as a subgroup of $\mathrm{Gal}(K/F)$. Prove that $\mathrm{Gal}(K/F) = \langle \mathrm{Gal}(K/L_1), \ldots, \mathrm{Gal}(K/L_n) \rangle$ if and only if $L_1 \cap \cdots \cap L_n = F$.

(5) What is the degree of the splitting field of $X^4 + 3$ over $\mathbb{Q}$?

(6) Let G be a p-group and $N \triangleleft G$ a nontrivial normal subgroup. Prove that $N \cap Z(G) \neq \{1\}$.