

ALGEBRA PRELIMINARY EXAM SEPTEMBER 2025

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such. If you do more than 3 problems, your score is based on the best 3.

PART 1: GROUPS, RINGS, MODULES, CATEGORY THEORY

- (1) Consider a short exact sequence of abelian groups

$$1 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{i} A \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

- (a) Find all possibilities for the isomorphism class of A .
 (b) For each possible isomorphism class of A decide if the exact sequence splits, and exhibit a splitting map if it does.

- (2) Let $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\} \subset \text{GL}_3(\mathbb{F}_p)$ be the group of 3×3

unipotent upper triangular matrices with entries over the field \mathbb{F}_p .

- (a) Compute the abelization $G^{ab} = G/[G, G]$ of G .
 (b) Show that the minimal number of generators needed to generate G is 2.

- (3) (a) Let D_{2n} be the group generated by two generators a, b and relations

$$b^2 = 1, a^n = 1, bab^{-1} = a^{-1}$$

Show that D_{2n} has order $2n$.

- (b) Let G be the group generated by two generators a, b and relations

$$b^2 = 1, bab^{-1} = a^{-1}$$

Show that G does not have finite order.

- (4) Show that the $\mathbb{C}[x]$ -module $\mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$ satisfies the descending chain condition but fails the ascending chain condition.

(Recall that an R -module M satisfies the descending (resp. ascending) chain condition if any sequence of submodules $M_1 \supset M_2 \supset \dots$ (resp. $M_1 \subset M_2 \subset \dots$) stabilizes).

- (5) Let Vect_k be the category of finite-dimensional vector spaces over a field k . Let $D : \text{Vect}_k \rightarrow \text{Vect}_k$ be the functor sending a vector space V to its dual vector space $D(V) = V^* = \text{Hom}_k(V, k)$.

- (a) Show that there are no natural transformation $T : D \Rightarrow D$ so that $T(V)$ is multiplication by 2 when $\dim_k V$ is even and $T(V)$ is multiplication by 3 when $\dim_k V$ is odd.
 (b) Describe all possible natural transformation $T : D \Rightarrow D$.

PART 2: LINEAR ALGEBRA AND GALOIS THEORY

- (1) Find all possible similarity classes of 4×4 matrices A with **minimal polynomial**

$$m_A(x) = (x - 2)^2(x + 1)$$

over

- (a) $k = \mathbb{C}$.
 (b) $k = \mathbb{F}_3$.
- (2) Let k be a field and let $D : k[x] \rightarrow k[x]$ be the derivative operator
- $$f(x) \mapsto f'(x)$$
- (a) Suppose $k = \mathbb{C}$. Show that the only k -vector subspaces of $k[x]$ which are preserved by D are the spaces P_n consisting of polynomials of degree $\leq n$ (for $n \geq 0$).
- (b) Suppose $k = \mathbb{F}_p$. Exhibit a subspace $V \subset \mathbb{F}_p[x]$ which is stable under D , but is not of the form P_n for any n .
- (3) Let \mathbb{F}_q be the finite field with q elements and let K be an extension of \mathbb{F}_q of degree n . Let $\alpha \in K$. Show that $\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}$ are linearly dependent over \mathbb{F}_q if and only if α is a root of a polynomial in $\mathbb{F}_q[x]$ of degree at most $n - 1$.
- (4) Consider the extension $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ obtained by adjoining an 8-th root of unity.
- (a) Compute the Galois group of this extension.
 (b) Describe all the quadratic subextensions of $\mathbb{Q}(\zeta_8)/\mathbb{Q}$.
- (5) Compute the Galois group of the polynomial $f(x) = x^4 + 9x^2 + 3$ over
- (a) $k = \mathbb{Q}$.
 (b) $k = \mathbb{F}_2$.

PART 3: COMMUTATIVE AND HOMOLOGICAL ALGEBRA

- (1) Let K be a field, and let $K \subset R \subset S$ where R and S are (commutative) K -algebras. Suppose that R is a finite K -algebra. (i.e. R is finitely generated as a K -module.)
 - (a) Suppose that $K = \mathbb{Q}$ and $S = \mathbb{C}$. Prove that R is a field.
 - (b) Give an example of general K, R, S where R need not be a field.
- (2) Let R be a ring. Prove that the topological space $\text{Spec } R$ is connected if and only if the only idempotents of R are 0 and 1.
- (3) Let $K = \mathbb{Q}[\sqrt{-3}]$.
 - (a) Let $\alpha \in K$. Prove that $\alpha \in K$ is integral over \mathbb{Z} if and only if its minimal polynomial has coefficients in \mathbb{Z} .
 - (b) Find (with proof) the set of elements of K that are integral over \mathbb{Z} .
- (4) (a) Prove Baer's criterion: An R -module I is injective when for every ideal $J \subset R$ and every map $f : J \rightarrow I$, there is a map $g : R \rightarrow I$ where $g|_J = f$.
 - (b) Show that \mathbb{Q}/\mathbb{Z} is an injective abelian group.
- (5) Let M be a finitely generated projective R -module. Prove that M is flat.