PRELIMINARY EXAM IN ALGEBRA FALL 2020

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Groups, rings, modules

Do **three** of the following five problems.

(1) Let G be a finite group with p-Sylow subgroup P for some prime p. Let X be the set of all p-Sylow subgroups. Then P acts on X by conjugation.
  (a) Prove that $Q \in X$ is fixed by this action if and only if $P = Q$.
  (b) Prove the number of p-Sylow subgroups is congruent to 1 mod p and divides the order of G.

(2) Let $R = \mathbb{Z}\sqrt{-5} = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$.
  (a) Prove that R is a Noetherian integral domain
  (b) Define $N : R \to \mathbb{Z}$ by $N(a + bi\sqrt{5}) = a^2 + 5b^2$. Show $N(xy) = N(x)N(y)$. Conclude that $x \in R$ is a unit if and only if $x = \pm 1$.
  (c) Consider the set of elements $A = \{2, 3, 1 \pm i5\}$. Show that every element of A is irreducible and that none is an associate of another. Show also that no element $x \in A$ generates a prime ideal.

(3) Let R be a commutative ring. Show that R is Noetherian if and only if every finitely generated R-module is Noetherian.

(4) Let R be a principal ideal domain. Prove it is integrally closed.

(5) Let k be a field and $R = k[x, y, z]$ be the polynomial ring in three variables. Consider the ideals $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, and $\mathfrak{m} = (x, y, z)$ of R.
  (a) Show $\mathfrak{p}_1$ and $\mathfrak{p}_1$ are prime ideals and $\mathfrak{m}$ is a maximal ideal.
  (b) Let $I = \mathfrak{p}_1\mathfrak{p}_2$. Show $I = (x^2, xy, xz, yz)$.
  (c) Show $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of I.

Part II. Fields, Galois theory, and representation theory

Do **three** of the following five problems.

(1) Suppose K/F is a normal algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

(2) Suppose L/K is a Galois extension of degree 225. Prove that there are intermediate fields $K \subsetneq K_1 \subsetneq K_2 \subsetneq L$ such that K_i/K is Galois for $i = 1, 2$.

(3) Suppose $\{t_1, \dots, t_n\}$ is a transcendence basis for K/k and that $p_1, \dots, p_n \in k[X]$ with all $\deg p_i \geq 1$. Prove that $\{p_1(t_1), \dots, p_n(t_n)\}$ is a transcendence basis for K/k.

(4) How many nonisomorphic 2-dimensional complex representations can a group of order 10 have?

(5) Suppose V and W are irreducible complex representations of finite groups G and H, respectively. Prove that $V \otimes W$ is an irreducible representation of $G \times H$ and moreover that every irreducible representation of $G \times H$ is of this form.

Part III. Linear and homological algebra

Do **three** of the following five problems.

(1) If K/k is a finite Galois extension, prove that $K \otimes_k K \cong K^{[K:k]}$ as k-algebras. Here A^B is the set of functions $A \to B$.

(2) (a) Let V be a finite dimensional vector space over an algebraically closed field and let $T : V \to V$ be a linear transformation. Suppose the minimal polynomial for T has distinct roots. Show T is diagonalizable.

(b) Suppose C_n is a cyclic group of order n and V is a finite dimensional representation of C_n over an algebraically closed field of characteristic prime n. Show that V is sum of one-dimensional representations.

(3) Let R be principal ideal domain and $a \in R$. Let M be an R-module.

(a) Show that

$$\begin{aligned} \mathrm{Hom}_R(R/a, M) &\cong \{x \in M \mid ax = 0\} \\ \mathrm{Ext}^1_R(R/a, M) &\cong M/aM \\ \mathrm{Ext}^s_R(R/a, M) &= 0, \quad s > 1. \end{aligned}$$

(b) Given an example of a principal ideal domain R, an element $a \in R$, and a non-zero R-module M so that

$$\mathrm{Hom}_R(R/a, M) = 0 = \mathrm{Ext}^1_R(R/x, M).$$

More Problems

(1) Give an example of a field extension K/k and a k-homomorphism $\phi : K \to K$ that is not an automorphism.

(2) Suppose V and W are two nonisomorphic, irreducible complex representations of a finite group G, and $T : V \to W$ is any $\mathbb{C}$-linear transformation. Prove that

$$\sum_{g \in G} g^{-1} T(gv) = 0$$