ALGEBRA PRELIMINARY EXAM, JUNE 2021

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such.

1. Part 1: Groups, Rings and Modules

(1) Show that a group of order 40 is solvable. Exhibit an example of a non-abelian group of order 40.

(2) Describe the conjugacy classes of S_5, and compute their sizes.

(3) Let M be a Noetherian module over a ring R and let $T : M \to M$ be an endomorphism.
   (a) Assume M is a Noetherian module. Show that if T is surjective, then T is an isomorphism.
   (b) Find an example to show that the previous item fails without the Noetherian assumption.

(4) (a) Show that the ring $\mathbb{Z}[\sqrt{-3}]$ is not a unique factorization domain.
   (b) Show that the ring $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is an Euclidean domain.
   (c) Determine the units of $\mathbb{Z}[\sqrt{-3}]$ and $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.

(5) Let R be a PID.
   (a) Characterize the finitely generated R-modules M such that $\mathrm{Hom}_R(M, R) = 0$.
   (b) Characterize the finitely generated R-modules M such that $\mathrm{Hom}_R(M, K/R)$ is finitely generated, where K is the fraction field of R.
   (c) Find an example of a non-finitely generated R-module M such that $\mathrm{Hom}_R(M, R)$ is finitely generated.

2. Part 2: Linear algebra and Galois theory

(1) (a) Find a complete set of representatives for the similarity classes of $A \in M_5(\mathbb{C})$ which has minimal polynomial $(x-1)^2(x+3)$.
   (b) Suppose $A \in M_n(\mathbb{C})$ such that A has minimal polynomial x^n. Show that any matrix B such that $AB = BA$ is of the form $f(A)$ for some $f \in \mathbb{C}[x]$.

(2) Let K be the splitting field of $x^3 - 3$ over $\mathbb{Q}$.
   (a) Describe the lattice of intermediate subfields of K.
   (b) Find a primitive element of K over $\mathbb{Q}$.

(3) Let V be an n-dimensional $\mathbb{C}$-vector space and let $A : V \to V$ be a linear endomorphism.
   (a) Show that A is nilpotent if and only if there is a filtration $Fil^0 = V \supseteq Fil^1 \supseteq \cdots Fil^n = 0$ such that $A(Fil^i) \subset Fil^{i+1}$.

(b) Define the commutator bracket $[S,T] = S \circ T - T \circ S$ for two endomorphisms S, T of V. Show that if A is nilpotent, then

$$[A,[A,[A,\cdots[A,T]\cdots] = 0$$

where there are $2n$ A's in the expression.

(4) (a) Show that $x^{p^n} - x$ is the product of all monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree dividing n. (you may use standard facts about the Galois theory of finite fields)

(b) Find the number of irreducible monic polynomials of degree 3 in $\mathbb{F}_3[x]$.

(5) Determine the Galois group of the polynomial $x^4 + 5x^2 + 5$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{5})$.

3. Part 3: Homological algebra, commutative algebra and representation theory

(1) Let M be a left module over a ring A. Prove that the following are equivalent:

(a) A is flat as a left A-module;

(b) $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is flat as a right A-module.

(2) Let H be the group of quaternionic units $\{\pm 1, \pm i, \pm j, \pm k\}$ that satisfy relations $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. List all conjugacy classes and all irreducible representations of H

i) over $\mathbb{R}$;

ii) over $\mathbb{C}$

together with their characters.

(3) Let F be a field. Is F^n an injective module over $M_n(F)$? Is it projective?

(4) Prove that a morphism in an Abelian category is a monomorphism if and only if its kernel (resp. cokernel) is zero. Give a counterexample in an additive category which is not Abelian.

(5) Give an example of a flat $\mathbb{Z}$-module N such that $\mathrm{Ext}^1_{\mathbf{Z}}(N, \mathbb{Z}) \neq 0$