PROPOSED SYLLABUS FOR ANALYSIS PRELIMS

1. Measure Theory

(1) Definition of Lebesgue measure $(\mathbb{R}^n, \mathcal{M}, m)$. Elementary sets, Lebesgue measurable sets and measurable functions. Outer measure. Borel measurability, continuous and semi-continuous functions. Basic properties of Lebesgue measures: monotonicity, regularity.

(2) Definition of the Lebesgue integral. Simple functions.

(3) The main convergence theorems: Monotone, Fatou, Lebesgue dominated convergence theorem.

(4) Distribution function. Chebychev inequality. Pushforward formula $f_*\mu$.

(5) Applications of Dominated convergence: differentiation under the integral sign, continuity of functions defined by integrals.

(6) $L^1(\mathbb{R}^n, dm)$ and more generally $L^p(X, \mu)$. Basic density theorems: simple functions, $C_c(\mathbb{R}^n)$.

(7) General measure spaces $(X, \mathcal{M}, \mu)$. Generalization of the simplest properties of Lebesgue measure. Carathéodory criterion for measurability. Integration. Fubini theorem.

(8) Egorov and Lusin theorems.

(9) Modes of convergence (pointwise a.e., L^1, in measure, uniform, almost uniform. Escape to horizontal infinity, escape to width infinity, escape to vertical infinity. Typewriter sequence.

(10) Absolute continuity of ν with respect to μ. $\epsilon - \delta$ test. Uniformly integrable sequence. Vitali convergence theorem.

(11) Fundamental theorem of Calculus 1: differentiating the integral. Lebesgue differentiation theorem on $\mathbb{R}^1$. Lebesgue differentiation theorem in $\mathbb{R}^n$.

(12) Fundamental theorem of Calculus 2: A.e differentiability of monotone functions. Functions of bounded variation. A.e. differentiability of BV functions. Absolutely continuous functions. Integrating the derivative.

Date: today.

References

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[Roy] H. L. Royden, Real analysis. Third edition. Macmillan Publishing Company, New York, 1988.

[R] W. Rudin, *Real and complex analysis.* Third edition. McGraw-Hill Book Co., New York, 1987.

[S] E. M. Stein and R. Shakarchi, *Real analysis. Measure theory, integration, and Hilbert spaces.* Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005

[Tao1] T. Tao, An introduction to measure theory. Graduate Studies in Mathematics, 126. American Mathematical Society, Providence, RI, 2011.

2. Functional Analysis

(1) Hilbert space (separable). Orthonormal basis. Parseval identity. Closed subspaces and orthogonal projection. $L^2(X,\mu)$. Completeness and density theorems. $L^2(S^1)$. Fourier series. Parseval theorem, Bessel inequality.

(2) Linear functionals and the Riesz representation theorem. Application: Radon-Nikodym theorem.

(3) Linear operators on a Hilbert space. Norm of an operator. Unitary or bounded self-adjoint operators. Schur-Young inequality.

(4) L^p spaces. Hölder and Minkowski inequalities. Minkowski inequality for integrals. Duality theorem. Weak convergence in L^p. $L^p(\mathbb{R}^n, dx)$. Convolution. Approximate identities and density theorems.

(5) Banach spaces. Hahn-Banach theorem. Baire Category and its application to bounded operators $T : X \to Y$ between Banach spaces. Open Mapping and Closed Graph theorems. Uniform boundedness principle. Applications: Convergence of Fourier series. Decay of Fourier coefficients.

(6) Compact sets in Hilbert space. Compact and Hilbert Schmidt operators. Spectral theorem for compact self-adjoint operators. Hilbert space method for solving constant coefficient PDE's on bounded domains in $\mathbb{R}^n$.

(7) Fourier transform as a unitary operator on $L^2(\mathbb{R}^n, dx)$. Plancherel and inversion formulae. Schwartz class functions. Fourier conjugation of differentiation and multiplication.

(8) $C(X)$ for a compact Hausdorff space. Compact sets in $C(X)$: Arzela-Ascoli theorem. Weierstrass density theorem and some of the Stone-Weierstrass theorem.

(9) Weak and weak* topologies. Banach-Alaoglu theorem.

(10) Radon measures: Positive linear functionals. Dual space of $C_0(X)$.

References

[E] L. C. Evans, Appendix to *Partial differential equations.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

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[S2] E. M. Stein and R. Shakarchi, *Functional Analysis.* Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005.

[Tao2] T. Tao, *An epsilon of room, I: real analysis. Pages from year three of a mathematical blog.* Graduate Studies in Mathematics, 117. American Mathematical Society, Providence, RI, 2010.

3. Complex Analysis

(1) The Cauchy Riemann equations. Laplace's equation. Analytic and harmonic functions.

(2) Isolated singularities of holomorphic functions. Taylor and Laurent expansions. Poles and essential singularities. Riemann's removable singularities theorem.

(3) Cauchy integral formula. Residue theorem. Calculation of integrals by residue methods. Mean value formula for harmonic functions.

(4) Growth of holomorphic functions. Maximum modulus principle. Cauchy estimates. Liouville theorem and its generalizations (e.g. analytic functions of polynomial growth). Open mapping theorem for holomorphic functions.

(5) Zeros of analytic function. Argument principle. Rouche's theorem. Jensen and Poisson-Jensen formulae. Zeros of polynomials.

(6) Normal families and uniform convergence on compact sets. Montel's theorem. Hurwitz's theorem. Compactness for uniformly bounded families of holomorphic functions.

(7) Holomorphic functions as conformal maps. Automorphisms of the plane, upper half-plane, disc, punctured disc, annulus. Schwarz Lemma and Schwarz-Pick Lemma.

(8) Riemann mapping theorem for simply connected domains $\neq \mathbb{C}$.

(9) Harmonic and subharmonic functions. Dirichlet boundary problem. Perron's method. Green's function and the Riemann mapping theorem.

References

[1] T. W. Gamelin, *Complex Analysis.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[2] R.E. Greene and S. G. Krantz, *Function theory of one complex variable.* Third edition. Graduate Studies in Mathematics, 40. American Mathematical Society, Providence, RI, 2006.

[3] E.M. Stein and R. Shakarchi, *Complex analysis.* Princeton Lectures in Analysis, II. Princeton University Press, Princeton, NJ, 2003