PRELIMINARY EXAM IN ANALYSIS SPRING 2021

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

Problem 1. This problem has two parts.

(a) State the three convergence theorems of Lebesgue integration theory: (1) the monotone convergence theorem; (2) Fatou's lemma; (3) the dominated convergence theorem.

(b) Let f be a nonnegative measurable function on a measure space $(X, \mathscr{F}, \mu)$. Show that

$$\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f}{n}\right) d\mu = \int_X f \, d\mu.$$

Problem 2. Suppose that $(X, \mathscr{F}, \mu)$ is a measure space such that $\mu(X) < \infty$ and f a measurable function such that $f > 0$ almost everywhere. Show that for any $\epsilon > 0$ there is a $\delta > 0$ such that $\int_E f \, d\mu \geq \delta$ for all $E \in \mathscr{F}$ such that $\mu(E) \geq \epsilon$.

Problem 3. Let f be a nonnegative measurable function. Then for $p \geq 1$,

$$\int_X f^p \, d\mu = p \int_0^\infty \lambda^{p-1} \mu\left\{f \geq \lambda\right\} d\lambda.$$

Problem 4. Let $[a, b]$ be a finite interval. (a) State the definition of a function of bounded variation on the interval $[a, b]$. Let f be an integrable function on $[a, b]$ and

$$F(x) = \int_a^x f(t) \, dt.$$

Show that the total variation of F on $[a, b]$ is given by

$$V(F)_a^b = \int_a^b |f(t)| \, dt.$$

Problem 5. Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ be two measure spaces and f a nonnegative measurable function on $(X \times Y, \mathscr{F} \times \mathscr{G})$. Let $1 \leq p \leq \infty$. Show that

$$\left\{\int_X \left[\int_Y f(x,y) \, \mu(dx)\right]^p \nu(dy)\right\}^{1/p} \leq \int_Y \left[\int_X f(x,y)^p \, \mu(dx)\right]^{1/p} \nu(dy).$$

Part II. Functional Analysis

Do **three** of the following five problems.

Problem 1. This problem has three parts.

(a) Let f and g be L^∞ functions with compact support. Show that the convolution $f * g \in C_c(\mathbb{R})$ (continuous functions of compact support). Hint: You might use that $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$).

(b) Let E be a bounded measurable subset of $\mathbb{R}$ and χ_E its characteristic function (also called indicator function). Let $-E = \{-x : x \in E\}$ and let $E - E = \{x - y : x, y \in E\}$. Show that $\chi_E * \chi_{-E} \in C_c(\mathbb{R})$ and satisfies $\chi_E * \chi_{-E}(0) = m(E)$, and $\chi_E * \chi_{-E}(x) = 0$ if $x \notin E - E$.

(c) Let F be a measurable subset of $\mathbb{R}$ such that $m(F) > 0$. Prove that $F - F$ contains some interval around 0.

Problem 2. Let $\mathcal{P}$ be the normed space of all polynomials on $\mathbb{R}$ equipped with the norm $||p|| = \max_j |\alpha_j|$ where α_j are the coefficients of $p = \sum_{j=1}^N \alpha_j x^j$ (where N is the degree of p). Is $(\mathcal{P}, ||\cdot||)$ a Banach space?

Problem 3. Let $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space) and assume that both (i) $f(n) = 0$ for all $n \in \mathbb{Z}$ and (ii) $\hat{f}(2\pi n) = 0$ for all $n \in \mathbb{Z}$. Prove or disprove that $f \equiv 0$ (Hint: Use the Poisson summation formula).

Problem 4. Let $T : H \to H$ be a bounded self-adjoint operator on an infinite dimensional Hilbert space.

(a) Define the point spectrum $\sigma_{pp}(T)$, the continuous spectrum $\sigma_c(T)$ and the residual spectrum $\sigma_r(T)$.

(b) Suppose that T is compact, injective and self-adjoint. Show that $0 \in \sigma_c(T)$ (i.e. that $\mathrm{Ran}(T)$ is dense in H).

(c) Suppose that T is bounded and self-ajdoint. Prove that

$$||T|| = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Problem 5. Let $T : X \to Y$ be a compact operator between Banach spaces. Prove that for any $\epsilon > 0$ there exists a finite dimensional subspace $M \subset \mathrm{Ran}(T)$ such that for all $x \in X$,

$$\inf_{m \in M} ||Tx - m|| \leqslant \epsilon ||x||.$$

Part III. Complex Analysis

Do **three** of the following five problems.

Problem 1. Let n be a positive integer and C the boundary of the unit disc centered at $z = 0$ and oriented in the counterclockwise direction.

(a) Compute the contour integral

$$\int_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z}.$$

(b) Use the result in (a) to evaluate the real-variable integral

$$\int_0^{2\pi} \sin^n(x)\, dx.$$

Problem 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R \in (0, \infty)$. Show that there exists a point w with $|w| = R$ such that $f(z)$ cannot be analytically continued to any open set which contains w.

Problem 3. Let $\mathcal{F}$ be a family of holomorphic functions on the open unit disk D. Suppose $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is a normal family and there exists a point $p \in D$ such that $\{f(p) : f \in \mathcal{F}\}$ is bounded. Prove $\mathcal{F}$ is a normal family.

Problem 4. An *entire transcendental function* is an entire function (that is, a holomorphic function $f : \mathbb{C} \to \mathbb{C}$) that is not a polynomial. Prove the following two assertions.

(a) Suppose $f(z)$ is an entire function. If there exist an integer $n \geqslant 1$ and a constant $C > 0$ such that $|f(z) \leqslant C|z|^n$ for all z, then $f(z)$ is a polynomial of degree less than or equal to n.

(b) An entire transcendental function comes arbitrarily close to every value $w \in \mathbb{C}$ outside every circle in the complex plane.

(b) Let $A_R = \{z : |z| > R\}$. If $f(z)$ is an entire transcendental function, then for every $R > 0$, $f(A_R)$ is dense in $\mathbb{C}$.

Problem 5. Determine the group of holomorphic bijections (automorphisms) of the set $|z| > 1$, the complement of the closed unit disk.