PRELIMINARY EXAM IN ANALYSIS SPRING 2022

Instructions:

(1) There are **three** parts to this exam: I (Measure Theory), II (Functional Analysis), and III (Complex Analysis). There are **five** problems in each part. You should present good solution to **three** problems from each part: if you present solutions to more than three problems in a part, the grader will select which three solutions contribute most to the total grade.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. If a problem asks you to state or prove a theorem or a formula, you need to provide the full details. If it asks you to disprove a statement, a counterexample will suffice, again of course with full details.

Part I. Measure Theory. λ denotes Lebesgue measure.

(1) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions such that $f_n \to f$ a.e.

(a) Show that for any $\epsilon > 0$ there exists a measurable set $A_\epsilon \subset [0,1]$ with $\lambda(A_\epsilon) < \epsilon$ so that f_n converges uniformly to f on $X \setminus A_\epsilon$.

(b) Does the above result hold if we replace $[0,1]$ by $\mathbb{R}$?

(2) (a) State Fatou's Lemma.

(b) Use Fatou's Lemma to prove the following. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f_n be a sequence of measurable functions such that $f_n \to f$ a.e. Assume there exists a sequence $g_n \in L^1$ such that $|f_n| \le g_n$, $g_n \to g$ a.e. and $g_n \to g$ in L^1. Show that

$$\lim_n \int f_n d\mu \to \int f d\mu.$$

(c) Does the above result hold if we assume $g_n \to g$ in measure instead of L^1 convergence?

(3) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$.

(a) Show that if $p < q \le \infty$ then $L^q \subset L^p$.

(b) Let $f \in L^\infty$. Prove that the function $p \to \|f\|_p$ is monotone and continuous.

(c) Establish the limit $\lim_{p\to 0} \log \|f\|_p = \int \log |f| d\mu$

(4) Suppose $f \in L^p(\mathbb{R})$, $1 \le p < \infty$. For $r \in \mathbb{R}$, set $T_r f(t) = f(t - r)$. Show that $T_r f \to f$ in L^p as $r \to 0$.

(5) For $f \in L^1(\mathbb{R})$ set for $x \in \mathbb{R}$

$$f^*(x) = \sup_{\epsilon>0} \left[\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} |f| d\lambda \right].$$

Show that for any $t > 0$

$$\lambda\left(\{x \in \mathbb{R} : f^*(x) \ge t\}\right) \le \frac{10\|f\|_1}{t}.$$

Part II. Functional Analysis. Notations: $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy,\ \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\langle x,\xi\rangle}$.

(1) Let $f \in L^2(\mathbb{R}, dx)$ with $||f|| = 1$ (all norms refer to L^2 norms). Let E resp. F be finite measure subsets of $\mathbb{R}$, and let $\mathbf{1}_E, \mathbf{1}_F$ be their indicator (characteristic) functions and define the projection operators $P_E f = \mathbf{1}_E f, \widehat{Q_F f} = \mathbf{1}_F \hat{f}$. Let m denote Lebesgue measure. Show that:

(a) Show that $||P_E \circ Q_F||_{L^2 \to L^2} \leq \sqrt{m(E)}\sqrt{m(F)}$.

(b) Show that $||P_E \circ Q_F||_{HS} = \sqrt{m(E)}\sqrt{m(F)}$.

(c) Suppose that $||(1 - \mathbf{1}_E)f|| < \epsilon_E$, and $||(1 - \mathbf{1}_F)\hat{f}|| < \epsilon_F$. Show that $m(E)m(F) \geq (1 - (\epsilon_E + \epsilon_F))$.

(2) Suppose that $S \subset L^2([0,1], dx)$ is a closed subspace. Suppose that $S \subset C[0,1]$ (continuous functions). Prove that S is finite dimensional.

(3) Let E be an infinite dimensional separable Hilbert space and let S be its unit sphere. Find the weak closure of S.

(4) Let $D \subset \mathbb{R}^2$ be the unit disk. Let $\mathbf{1}_D$ be the indicator function of D.

(a) Find the largest $s \in \mathbb{R}$ so that $\mathbf{1}_D \in H^s(\mathbb{R}^2)$.

(b) Let δ_{S^1} be the measure $\delta_{S^1}(f) = \int_{S^1} f d\theta$ where $S^1 = \partial D$ is the unit circle and $f \in C(S^1)$. Find the largest s so that $\delta_{S^1} \in H^s(\mathbb{R}^2)$. Find the smallest t so that $\delta_{S^1}(f)$ is a bounded linear functional on $H^t(\mathbb{R}^2)$.

(5) Let $f \in L^1(\mathbb{R}^n)$ and let $T_f g = f * g$ for $g \in L^2(\mathbb{R}^n)$.

(a) Prove the following special case of Young's convolution inequality: $||T_f g||_{L^2} \leq ||f||_{L^1}||g||_{L^2}$.

(b) Show that the above inequality is sharp by showing it is achieved if f and g are Gaussian functions $f = g = e^{-x^2/2}$.

(c) Find $||T_f||_{L^2 \to L^2}$.

(d) Compute the spectrum of T_f and say which subset consists of continuous spectrum and which subset consists of discrete spectrum.

Part III. Complex Analysis. Below, the terms 'complex analytic, holomorphic' are all synonyms.

(1) Consider the analytic polynomial $p_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$. Let
$$R = \sqrt{|a_0|^2 + \cdots + |a_{n-1}|^2 + 1}$$
and assume $R > 1$. How many zeros of $p_n(z)$ are contained in the open disc $D(0, R)$ centered at 0 of radius R?

(2) Suppose that $p(x)$ is a positive polynomial (hence, has no real zeros). Compute $\int_{-\infty}^{\infty} \frac{dx}{p(x)}$.

(3) Let $\mathbb{D} = \{z : |z| < 1\}$. If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and injective and $f'(0) = 1$, prove that the area of $f(\mathbb{D})$ is at least π.

(4) Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$. Suppose that
$$\int_{\mathbb{D}} |f_n(z)| dx dy \leq 1, \quad \forall n.$$
Show that $\{f_n\}$ is a normal family.

(5) Let f be a non-constant entire function.

(a) Show that the range of $f := f(\mathbb{C})$ is dense in $\mathbb{C}$.

(b) Show that the range of $f := f(\mathbb{C})$ cannot omit any half-line (i.e. cannot be contained in the complement of the half-line.)