Preliminary Exam in Analysis
June 2026

Instructions:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space.
  (a) State the Monotone Convergence Theorem and the Dominated Convergence Theorem.
  (b) Prove that if $f : X \to \mathbb{R}$ is nonnegative and measurable then

$$\lim_{n\to\infty} \int_X \frac{nf}{n+f} d\mu = \int_X f d\mu.$$

(2) Let $f \in L^1(\mathbb{R})$ and define

$$I(t) = \int_{\mathbb{R}} |f(x+t) - f(x)| \, dx.$$

Prove that $I(t) \to 0$ as $t \to 0$. *(Hint: first consider the case when $f(x) = 1_{[a,b]}(x)$ is the indicator function of an interval.)*

(3) Fix $1 \le p < \infty$. Show that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\lambda} \int_0^\varepsilon f dx = 0 \quad \text{for all } f \in L^p([0,1]),$$

if and only if $\lambda \le 1 - 1/p$.

(4) Let $f \in L^1(\mathbb{R})$ and suppose that

$$F(a) := \int_a^{a+1} f(x) dx = 0$$

for every $a \in \mathbb{R}$. Show that $f = 0$ almost everywhere.

(5) Let X be a set.
  (a) Define what it means for $\mu_* : \mathcal{P}(X) \to [0, \infty]$ to be an *exterior measure on* X, where $\mathcal{P}(X)$ is the collection of all subsets of X.
  (b) Define what it means for a subset E of X to be *Carathéodory measurable* with respect to μ_*.
  (c) Show directly from your definitions that if E_1 and E_2 are Carathéodory measurable then so is $E_1 \cup E_2$.
  (d) Show that if E_1, E_2 are Carathéodory measurable and disjoint then $\mu_*(E_1 \cup E_2) = \mu_*(E_1) + \mu_*(E_2)$.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let f be a measurable function on $\mathbb{R}$. Suppose that there is an index $1 \le p \le \infty$ and a constant C such that fg is integrable for all $g \in L^p(\mathbb{R})$ and

$$\left| \int_{\mathbb{R}} f(x)g(x)\,dx \right| \le C\|g\|_p.$$

Show that $f \in L^q(\mathbb{R})$ and $\|f\|_q \le C$. Here q is the conjugate index given by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(2) Let $H = L^2[0,1]$ be the Hilbert space of square integrable functions on $[0,1]$. Define the integral operator

$$Kf(x) = \int_0^x f(y)\,dy, \quad f \in H.$$

(a) Show that K is a compact operator.
(b) Find the adjoint operator K^*.
(c) Find the integral kernel of the composition $L = K^*K$; namely, find the function $L(x,y)$ such that

$$Lf(x) = \int_0^1 L(x,y)f(y)\,dy.$$

(3) (a) Let Y be a proper closed subspace of a Banach space X and $0 < \epsilon < 1$. Show that there is an element $x \in X$ such that $\|x\| = 1$ and $\|x - y\| \ge 1 - \epsilon$ for all $y \in Y$.
(b) Show that every locally compact Banach space must be finite dimensional.

(4) Consider the indicator function $f = 1_{[-1,1]}$ of the interval $[-1,1]$ on $\mathbb{R}$.
(a) Compute the Fourier transform $\hat{f}$ of the function f.
(b) For what values of s does f lie in $H^s(\mathbb{R})$?

(5) Let $f \in L^2(\mathbb{R})$ and $\hat{f}$ its Fourier transform. Suppose that

$$\int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2\,d\xi < \infty.$$

Show that f is equal almost everywhere to a function h which is absolutely continuous, with $h' \in L^2(\mathbb{R})$.

Part III. Complex Analysis

In the following questions, $D_a(w)$ denotes the open disc of radius a centered at w and $\overline{D}_a(w)$ its closure.

Do **three** of the following five problems.

(1) Show that for any $\epsilon > 0$ the function

$$\frac{1}{z+i} + \sin z$$

has infinitely many zeros in the strip $|\mathrm{Im}\, z| < \epsilon$.

(2) Let

$$\Omega = \mathbb{C} \backslash ((-\infty, -1] \cup [1, \infty)).$$

Find a biholomorphism of Ω to $D_1(0)$.

(3) Show that there exists no function f holomorphic on a neighborhood of $\overline{D}_1(0)$ such that

$$\sup_{\theta \in [0,2\pi]} \left| f(e^{i\theta}) - \sin(3\theta) \right| < \frac{1}{100}.$$

(4) For $0 < r < 1$, evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}.$$

(5) Recall that the Riemann mapping theorem says that for any nonempty simply connected domain $\Omega \subset \mathbb{C}$ with $\Omega \neq \mathbb{C}$, for any choice of $z_0 \in \Omega$, there exists a unique biholomorphism

$$f : \Omega \to D_1(0)$$

with $f(z_0) = 0$, $f'(z_0) \in (0, \infty)$.

(a) Prove that the restriction $\Omega \neq \mathbb{C}$ is necessary for the theorem to hold.

(b) Prove the uniqueness part of the theorem using the Schwarz lemma.