PRELIMINARY EXAM IN ANALYSIS JUNE 2018

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State the Monotone Convergence Theorem for a σ-finite measure space $(X, \mathcal{M}, \mu)$.

(b) Show that if $\{f_n\}$ is an increasing sequence of **nonpositive** (i.e. $f_n \leq 0$) measurable functions with $\int_X |f_1| < \infty$ then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu.$$

(c) Show that the conclusion of (b) is false in general if the assumption $\int_X |f_1| < \infty$ is dropped.

(2) Let f be a continuous function on $[0,1]$ and show that:

$$\lim_{n\to\infty} n \int_0^1 x^n f(x) dx = f(1).$$

(3) Let E be a Lebesgue measurable set in $\mathbb{R}^d$ whose Lebesgue measure is finite. Fix $p > 1$ and consider a sequence $\{f_n\}$ of functions in $L^p(E)$. Consider the following notions of convergence:

(i) $f_n \to f$ (strongly) in $L^p(E)$.

(ii) $f_n \to f$ (strongly) in $L^1(E)$.

(iii) $f_n \to f$ in measure on E.

(iv) $f_n \to f$ pointwise a.e. on E.

Show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Does (iii) $\Rightarrow$ (iv)?

(4) Let E be a Lebesgue measurable subset of $[0,1]$ of positive measure. Show that there exist $k, n \in \mathbb{N}$ and $x, y \in E$ satisfying $|x - y| = k/2^n$.

Hint. Argue by contradiction and consider the sets $E + \frac{1}{2^n}$.

(5) (a) Define the *total variation* $T_f(a,b)$ of a function $f : [a,b] \to \mathbb{R}$.

(b) Show that if f is continuous on $[a,b]$ and continuously differentiable on (a,b) then $T_f(a,b) \leq \int_a^b |f'|dx$.

(c) Show that if $r > s > 0$ then the function

$$f(x) = \begin{cases} x^r \sin(x^{-s}), & \text{for } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

has bounded variation.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $T : H \to H$ be a non-trivial compact operator on an infinite dimensional Hilbert space. Let $0 \neq \lambda \in \mathbb{C}$.

(a) Show that if λ is not an eigenvalue of T then there exists $C > 0$ so that $||(T-\lambda)x|| \geq C||x||$ for all $x \in H$.

(b) Show that $(T-\lambda) : H \to H$ is surjective.

(c) Conclude from (A)-(B) that $(T-\lambda)^{-1}$ is a bounded operator on H.

(2) Let $\phi \in C[0,1]$ and let $M_\phi f = \phi f$ be the corresponding multiplication operator on $C[0,1]$.

(a) Prove: Either M_ϕ is surjective or the range $M_\phi C[0,1]$ has first Baire category in $C[0,1]$.

(b) Find a necessary and sufficient condition that $M_\phi C[0,1]$ is of first Baire category.

(3) (a) Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Prove that $f * g \in \mathcal{S}(\mathbb{R}^n)$.

(b) Let $f \in L^2(\mathbb{R}^n)$ and $\phi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Let $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$. Prove that $f * \phi_\epsilon \to f$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$.

(4) Suppose that $f \in L^p(X,\mu)$ for some $p > 0$ and that $\mu(X) = 1$. Show that when $\log|f| \in L^1$ then $\lim_{q \to 0_+} ||f||_q = \exp\left(\int_X \log|f| d\mu\right)$.

(5) Let $f \in L^1[0,1]$ but $f \notin L^2[0,1]$.

(a) Prove that the subspace $\{\psi \in L^2[0,1] : \int f\psi = 0\}$ is dense in $L^2[0,1]$.

(b) Show that if S is a dense subspace of a Hilbert space H, then there exists an orthonormal basis H consisting of elements of S.

(c) Conclude that there exists an orthonormal basis $\{\phi_n\}$ of $L^2[0,1]$ so that $\phi_n \in C([0,1])$ and $\int_0^1 f\phi_n dx = 0$ for all n.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Find explicitly a Riemann map (that is, a biholomophic bijection) of the open unit disk D onto each of the following domains:
   (a) the upper half-plane minus a slit $\{z \in \mathbb{C} : \mathrm{Im} z > 0\} \setminus \{z = it, 0 < t \leq T\}$,
   (b) the strip $\{z \in \mathbb{C} : -1 < \mathrm{Im} z < 1\}$.

(2) In each item, determine all entire functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy
   (a) $f(z+1) = f(z), \quad |f(z)| \leq e^{|z|}$.
   (b) $f(z+1) = f(z), \quad f(z+1+i) = f(z)$.

(3) Let f be a holomorphic function in the unit disc D with $f(0) = f'(0) = 0$. Show that $g(z) = \sum_{n=1}^{\infty} f(\frac{z}{n})$ defines an analytic function on D. Show that $g(z) = cf(z)$ for some constant $c \in \mathbb{R}$ if and only if $f(z) = \alpha z^2$ for some $\alpha \in \mathbb{C}$.

(4) Let $A := \{z \in \mathbb{C} : -1 < \mathrm{Im} z < 1\}$. Consider the set of functions defined by

$$\mathcal{S} = \{f : A \to \mathbb{C} | f \text{ holomorphic, } f(0) = 0, \text{ and } |f(z)| < 1\}.$$

Prove that

$$\sup_{f \in \mathcal{S}} |f(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.$$

(5) Let $P(z)$ be a polynomial all of whose zeros lie in $\{z : \mathrm{Re} z \leq 0\}$. Prove that $P'(z) \neq 0$ if $\mathrm{Re} z > 0$. Also, show that if $P'(z) = 0$ for z with $\mathrm{Re} z = 0$, then either z is a zero of order at least two of P or all zeros of P are on the imaginary axis.