PRELIMINARY EXAM IN ANALYSIS FALL 2018

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State the Dominated Convergence Theorem for Lebesgue measurable functions.

(b) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$ satisfying

$$0 \leq f_n(x) \leq \sin^{2n}(\pi x), \quad 0 \leq x \leq 1.$$

Show that $\int_0^1 f_n dx \to 0$ as $n \to \infty$.

(c) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $\mathbb{R}$ satisfying

$$0 \leq f_n(x) \leq \sin^{2n}(\pi x), \quad x \in \mathbb{R}.$$

Is it true that $\int f_n dx \to 0$ as $n \to \infty$? Explain your answer.

(2) Let $E \subset \mathbb{R}$ be Lebesgue measurable. For a fixed $c \in (0,1)$ suppose that

$$m(E \cap (a,b)) \leq c(b-a)$$

for every $a, b \in \mathbb{R}$ with $a < b$. Show that $m(E) = 0$.

Note: You may use without proof the fact that every open set in $\mathbb{R}$ can be written as the countable disjoint union of open intervals.

(3) Let $0 < r < p < q < \infty$. For $f \geq 0$ in $L^p(\mathbb{R})$ show that $f = g + h$ where $g \in L^r(\mathbb{R})$ and $h \in L^q(\mathbb{R})$. Moreover, show that given $N > 0$, g and h can be chosen so that

$$\|g\|_r^r \leq N^{r-p}\|f\|_p^p, \quad \text{and} \quad \|h\|_q^q \leq N^{q-p}\|f\|_p^p.$$

Hint: consider the sets $\{f > N\}$ and $\{f \leq N\}$.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Show that if $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ satisfies $E_{k+1} \subset E_k$ for all k and $\mu(E_1) < \infty$ then

$$\mu(E) = \lim_{k\to\infty} \mu(E_k).$$

where $E = \bigcap_{k=1}^{\infty} E_k$.

(b) Let ν be a **finite** measure on $(X, \mathcal{M})$ which is absolutely continuous with respect to μ (namely $\nu(E) = 0$ whenever $E \in \mathcal{M}$ with $\mu(E) = 0$). Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ with $E \in \mathcal{M}$ implies $\nu(E) < \varepsilon$.

Hint: argue by contradiction and let $A_n \in \mathcal{M}$ satisfy $\mu(A_n) < 2^{-n}$ and $\nu(A_n) \geq \varepsilon$. Consider the decreasing sequence $E_k = \cup_{n\geq k} A_n$.

(5) (a) Define what it means for a function $F : [a, b] \to \mathbb{R}$ to be absolutely continuous.

(b) Suppose that $F : [0, 1] \to \mathbb{R}$ has the following properties:
   (i) For every $\varepsilon > 0$, F is absolutely continuous on $[\varepsilon, 1]$.
   (iii) $F = G - H$ where $G, H : [0, 1] \to \mathbb{R}$ are increasing functions which are continuous at 0.

Show that F is absolutely continuous on $[0, 1]$.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space and let $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$ be the Fourier transform of f. Define the 'periodization operator' $\mathcal{P} : \mathcal{S}(\mathbb{R}) \to C(S^1)$ by $Pf(x) = 2\pi \sum_{n\in\mathbb{Z}} f(x + 2n\pi)$.Here, $S^1 \simeq [0, 2\pi]/(0 \sim 2\pi)$ is the unit circle $\mathbb{R}/2\pi\mathbb{Z}$.

- (i) Show that $\mathcal{P}f \in C(S^1)$ if $f \in \mathcal{S}(\mathbb{R})$ (equivalently, that f is a periodic continuous function on $\mathbb{R}$ of period 2π).
- (ii) There is a second way to periodize (make periodic) $f \in \mathcal{S}(\mathbb{R})$: Let $\mathcal{Q}f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$. Show that $\mathcal{P}f(x) = \mathcal{Q}f(x)$.

(2) Let $L^p(X, \mu)$ be the L^p space of a measure space.

- Define 'weak convergence in L^p'.
- Show that the L^p norm is weakly lower semi-continuous: If $f_j \rightharpoonup f$ (i.e. f_j tends to f weakly in L^p), then

$$\liminf_{j\to\infty} ||f_j||_p \geq ||f||_p.$$

(Hint: recall the map $f \to |f|^{p-2}\bar{f}$.)

(3) Let T be the operator on $L^2[[0,1],dx]$ defined by

$$Tf(x) = \int_0^x f(y)dy.$$

- (i) Show that T is compact.
- (ii) Show that 0 is in the spectrum of T, and in fact is in the spectrum of every compact operator on an infinite dimensional Hilbert space. (The spectrum of a bounded operator T is $\{\lambda \in C : (T-\lambda)$ is not invertible as a bounded operator).
- (iii) Show that T has no eigenvalues.

(4) Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be a bounded operator on H. Let $\mathrm{Ran(T)}$ be the range of T. Let T^* be the adjoint of T.

- (i) Show that $H = \ker T \oplus \overline{\mathrm{RanT^*}}$ where $\oplus$ is orthogonal direct sum.
- (ii) Give an example of a bounded operator T on $L^2[0,1]$ such that $\mathrm{Ran(T)}$ is not closed (with proof).
- (ii) Suppose that there exists $C > 0$ so that $||f|| \leq C||Tf||$ for all f. Show that $\mathrm{Ran(T)}$ is closed.

(5) Let U be a unitary operator on a Hilbert space H. Let

$$Pf := s - \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} U^k f,$$

where $s - \lim$ means the limit in L^2 (strong limit).

- (i) Let $H_U = \{v \in H : Uv = v\}$. Let $W = \{Uv - v : v \in H\}$. Show that H_U is a closed subspace, that $H_U \perp W$ and that $P : W \to \{0\}$..
- (ii) Show that $H = \overline{H_U \oplus W}$. (Hint: If $f \perp W$, consider $f - Uf$.)
- (iii) Show that the limit exists for every $f \in H$ and that P is an orthogonal projection. Onto what subspace?

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Without using Picard's theorem, show the following.
  (a) Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense.
  (b) Show that if f is entire and if there is a line L such that $f(\mathbb{C}) \cap L = \emptyset$ then f is constant.

(2) Show that for every $\lambda > 1$, the equation $e^z - z = \lambda$ has exactly one solution in the half-plane $Re(z) < 0$ and this solution is real.

(3) Let $\Omega \subset \mathbb{C}$ be a bounded domain and $\{f_j\}$ be a sequence of holomorphic functions on Ω. Assume

$$\int |f_j(z)|^2 dz < C < \infty$$

for some C that does not depend on j. Show that $\{f_j\}$ is a normal family, that is, every subsequence of $\{f_j\}$ has a convergent subsequence that converges uniformly on compact sets of Ω.

(4) Determine all complex analytic functions f on the unit disc which satisfy

$$f''\left(\frac{1}{n}\right) + \pi f\left(\frac{1}{n}\right) = 0$$

for $n = 2, 3, 4, \ldots$

(5) (a) Let f be an entire function that satisfies $\lim_{z\to\infty} |f(z)| = \infty$. Show that f is a polynomial.

(b) Let f and g be entire functions such that

$$\lim_{z\to\infty} f(g(z)) = \infty.$$

Show that both f and g are polynomials.