PRELIMINARY EXAM IN ANALYSIS SEPTEMBER 2022

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Let $r \geq 0$ and $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^{nr} \cos^4(nx) \exp(x/n)$. For which values of $r \geq 0$ does the integral $\int_0^1 f_n(x)dx$ converge as $n \to \infty$? Justify your answer.

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$ and $f_n, f : X \to \mathbb{R}$ measurable functions. Show that $f_n \to f$ almost everywhere if and only if

$$\sup_{k \geq n} |f_k - f| \to 0 \text{ in measure as } n \to \infty.$$

(3) Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose f_n is a sequence of nonnegative integrable functions on X and A a constant such that for any measurable set E and any $n \geq 1$

$$\int_E f_n(x) d\mu \leq A\mu(E)$$

Assume that $f_n \to f$ almost surely. Does it hold that $\int_X f_n d\mu \to \int_X f d\mu$? What if $\mu(X) < \infty$?

(4) (a) State Fubini's Theorem.

(b) Using Fubini's Theorem, prove that

$$\lim_{M \to \infty} \int_0^M \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

HINT: Use $x^{-1} = \int_0^\infty e^{-xt} dt$.

(c) Is the function $\frac{\sin x}{x}$ in $L^1(\mathbb{R})$? Justify your answer.

(5) (a) State the Lebesgue differentiation theorem for continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(b) Let $f : \mathbb{R} \to R$ be an integrable function. Using the Theorem stated in part 5 (a), show that

$$\lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} f(y) dy = f(x)$$

for almost every $x \in \mathbb{R}$.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) (a) Does there exist a non-zero Schwartz function $g \in \mathscr{S}(\mathbb{R})$ such that the convolution operator T_g is a compact operator on $L^2(\mathbb{R})$? Prove that your answer is correct.

(b) Does there exist a non-zero C^∞ function $g \in C^\infty(\mathbb{S}^1)$ such that the convolution operator T_g is a compact operator on $L^2(\mathbb{S}^1)$? Prove that your answer is correct.

Recall that the convolution operator T_g on $\mathbb{R}$ is defined by

$$T_g f(x) = (f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy, \ \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\langle x,\xi\rangle},$$

and the convolution operator on the unit circle $\mathbb{S}^1$ is defined by

$$T_g f(e^{i\theta}) = (f * g)(e^{i\theta}) = \int_0^{2\pi} f(e^{i(\theta-\phi)}g(e^{i\phi})d\phi.$$

(2) Let $(X, ||\cdot||)$ be an infinite dimensional separable Banach space. Can it be spanned, as a vector space, by countably many elements $\{x_j\}_{j=1}^\infty$? I.e. can every element of X be expressed as a finite linear combination of the x_j? Prove that your answer is correct.

(3) Let B be a Banach space and let $S \subset B$ be a closed subspace. Let $f_0 \in B, f_0 \notin S$. Show that there exists $\ell \in B^*$ so that $\ell|_S = 0$ and $\ell(f_0) = \mathrm{dist}(f_0, S)$ (the distance in B from f_0 to S.) Deduce that a linear functional on B is bounded (i.e., belongs to B^*) if and only if its kernel is a closed subspace.

(4) Find all $f \in \mathscr{S}(\mathbb{R})$ (Schwartz space) with the property that its Fourier transform $\hat{f}$ is zero on $2\pi\mathbb{Z}$.

(5) Let $M \subset L^2([0,1], dx)$ be a closed subspace such that $M \subset C([0,1])$. Show that,

(a) There exists $C > 0$ so that $||f||_\infty \le C||f||_2$ for $f \in M$.

(b) Show that for all $x \in [0,1]$ there exists $g_x \in M$ so that for $f \in M$, $f(x) = \langle f, g_x\rangle_{L^2}$ and $||g_x||_{L^2} \le C$.

(c) Show that $\dim M \le C^2$.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $\mathcal{P}_{\mathbb{Z}} = \{f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{Z}, a_n \neq 0$ for an infinite number of $n\}$ be the set of power series with an infinite number of non-zero coefficients, all of whose coefficients are integers. What is the maximal radius of convergence of $f \in \mathcal{P}_{\mathbb{Z}}$? Prove that your answer is correct.

(2) Let $0 < R < \frac{\pi}{2}$. Prove that for sufficiently large n, the polynomial

$$P_n(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!}$$

has no roots with modulus $< R$.

(3) Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and that it is injective in some annulus $\{z : r < |z| < 1\}$. Prove that f is injective on all of $\mathbb{D}$.

(4) Answer the following two questions:

(a) Is the punctured plane $\mathbb{C}\backslash\{0\}$ conformally equivalent (i.e. biholomorphic) to the punctured unit disc $D\backslash\{0\}$ ($D = \{z : |z| < 1\}$. Prove that your answer is correct.

(b) Is the horizontal strip $\{0 < \Im z < 1\}$ conformally equivalent to the unit disc? If not, prove it. If it is, construct a conformal equivalence.

(5) Let f be an entire complex analytic function and let $\{|f| = c\}$ with $c > 0$ be a level set of its modulus. Assume that c is a regular value of $|f|$, i.e. , every component of the level set is a smooth simple closed curve. Prove that every connected component of the domain $\{z : |f(z)| < c\}$ must contain a zero (i.e., a point z_0 such that $f(z_0) = 0\}$