PRELIMINARY EXAM IN ANALYSIS
SEPTEMBER 2025

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State Monotone Convergence Theorem.
  (b) State Fatou's Lemma.
  (c) Assume Fatou's Lemma and deduce the Monotone Convergence Theorem from it.

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be a measurable function from X to $[0, \infty]$. Define the measure λ by $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Show that for any measurable function g from X to $[0, \infty]$, $\int_X g \, d\lambda = \int_X fg \, d\mu$.

(3) Let $X = Y$ be the interval $[0, 1]$, $\mathcal{M} = \mathcal{N}$ the Borel σ-algebra on $[0, 1]$, $\mu =$ Lebesgue measure, and $\nu =$ counting measure. Let $D = \{(x, x) : x \in [0, 1]\}$ be the diagonal in $X \times Y$ and χ_D the characteristic function of D. Compute $\int \int \chi_D \, d\mu \, d\nu$, $\int \int \chi_D \, d\nu \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$.
  (Hint: To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

(4) Suppose ν is a signed measure and λ, μ are positive finite measures on $(X, \mathcal{M})$ such that $\nu = \lambda - \mu$. Show that $\lambda \geq \nu^+$ and $\mu \geq \nu^-$, where ν^+, ν^- are positive and negative variations of ν.

(5) Let F be an increasing function on $\mathbb{R}$. Show that $F(b) - F(a) \geq \int_a^b F'(t) \, dt$.

Part II. Functional Analysis

Note: You may use any (consistent) normalization that you prefer for Fourier transforms and Fourier series. Do **three** of the following five problems.

(1) Let $1 < p, q < \infty$ be a pair of conjugate indices, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Suppose that $f(x, y)$ is a nonnegative Borel measurable function on $\mathbb{R}^2$. For each $y \in \mathbb{R}$, define the nonnegative Borel measurable function f_y on $\mathbb{R}$ by $f_y(x) = f(x, y)$. Its L_p-norm is

$$\|f_y\|_p = \left[\int_{\mathbb{R}} f(x, y)^p \, dx\right]^{1/p}.$$

(a) Let h be nonnegative Borel measurable function on $\mathbb{R}$. Show by Hölder's inequality that

$$\int_{\mathbb{R}} h(x) \left[\int_{\mathbb{R}} f_y(x) \, dy\right] dx \leq \|h\|_q \int_{\mathbb{R}} \|f_y\|_p \, dy.$$

(b) Use part (a) to show the following continuous version of the Minkowski inequality:

$$\left\| \int_{\mathbb{R}} f_y \, dy \right\|_p \leq \int_{\mathbb{R}} \|f_y\|_p \, dy.$$

(2) Let X be a Banach space and $\{x_n\}$ a sequence in X converging weakly to $x \in X$.

(a) Show that $\{x_n\}$ is norm bounded, i.e., there is a constant C such that $\|x_n\| \leq C$ for all n;

(b) Show that $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$;

(c) Give an example showing that the inequality in part (b) can be strict.

(3) Let H be a Hilbert space and $A : H \to H$ a linear operator defined on the whole space H such that $(Ax, y) = (x, Ay)$ for all $x, y \in H$. Show that A must be a bounded linear operator.

(4) Show that every (algebraically) finite dimensional subspace Y of a Banach space X must be norm closed; namely, if $y_n \in Y$ and $\|y_n - y\| \to 0$ as $n \to \infty$ for some $y \in X$, then $y \in Y$.

(5) Suppose that $f \in L^2(\mathbb{R})$ such that $\xi \hat{f}(\xi)$ is integrable, where $\hat{f}$ is the Fourier transform of f. Show that f is equal to a continuously differentiable function almost everywhere.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Suppose that $f : \mathbb{D} \to \mathbb{D}$ is not identically zero, it is holomorphic on the unit disk, and extends continuously to $\overline{\mathbb{D}}$. Let $a_1, a_2, \ldots$ be the zeros of f (repeated with multiplicity, and could be a finite or infinite set). Show that

$$|f(0)| \leq \prod_k |a_k|.$$

(2) Let $f : \Omega \to \mathbb{C}$ be holomorphic, where $\Omega = \mathbb{D} \setminus \{0\}$ is the punctured unit disk. Suppose that there is a constant $C > 0$ such that $|f(z)| \leq C|z|^{-1/2}$ for all $z \in \mathbb{D}$. Show that f has a removable singularity at $z = 0$.

(3) Use contour integration to compute the integral

$$\int_0^\infty \frac{\log x}{x^2 + 1}\, dx.$$

(4) Let Ω_1 be the half infinite strip $\{z : \operatorname{Re} z \in (-1, 1), \operatorname{Im} z > 0\}$. Let Ω_2 be the infinite strip $\{z : \operatorname{Im} z \in (0, 1)\}$. Find a biholomorphism $f : \Omega_1 \to \Omega_2$.

(5) Let $f(z) = e^z + z$. Find the number of zeros of f in the disk $|z| < 1/2$.