ANALYSIS PRELIMINARY EXAM
MONDAY, JUNE 13, 2011

Part I. Do three of the following five problems.

(1) (a) State the dominated convergence theorem and Fatou's lemma.
(b) Show that the inequality in Fatou's lemma may be strict.
(c) Use Fatou's lemma to prove the dominated convergence theorem.

(2) Suppose that $(X, \mathscr{F}, \mu)$ is a measure space with $\mu(X) = 1$. Let $f : X \to \mathbb{R}$ be a measurable function such that $f \in L^p(X, \mathscr{F}, \mu)$ for all $p \geq 1$. Suppose that f is not equal to a constant almost everywhere. Define $\phi(p) = \|f\|_p$, the p-norm of f. Show that ϕ is a strictly increasing function on $[1, \infty)$.

(3) Let $f \in L^p(X, \mathscr{F}, \mu)$ be a nonnegative L^p-integrable function on a measure space $(X, \mathscr{F}, \mu)$. Show that for any $p \geq 1$,

$$\int_X f(x)^p \, dx = p \int_0^\infty \lambda^{p-1} \mu \{f \geq \lambda\} \, d\lambda.$$

Here $\mu \{f \geq \lambda\}$ is the measure of the set $\{x \in X : f(x) \geq \lambda\}$.

(4) Suppose that $f : [0,1] \to \mathbb{R}_+$ is a nonnegative Lebesgue measurable function on the unit interval $[0,1]$ such that $f > 0$ almost everywhere. Show that for any $\epsilon > 0$, there is a $\delta > 0$ with the following property: if E is a measurable subset of $[0,1]$ with Lebesgue measure $m(E) \geq \epsilon$, then

$$\int_E f(x) \, dx \geq \delta.$$

(5) Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $V(f)_a^x$ the total variation of f on the interval $[a, x]$. Suppose that f is continuous at a point $c \in (a, b)$. Show that $V(f)_a$ is also continuous at $x = c$.

Part II. Do three of the following five problems.

(1) Let $L^1[0,1]$ be the Banach space of real-valued, Lebesgue integrable functions on the unit interval $[0,1]$ with the usual norm.
(a) Identify (with proof) the dual space $L^1[0,1]^*$.
(b) Is the unit ball in $L^1[0,1]$ weakly compact? Prove your answer is correct.

(2) Suppose that A is a linear operator defined everywhere on a Hilbert space H satisfying $\langle Av, w \rangle = \langle v, Aw \rangle$ for all $v, w \in H$. Show that A is bounded.

(3) We use $\mu(f)$ to denote the integral of a function f with respect to a measure μ. Denote the Lebesgue measure on $[0,1]$ by m. Find a sequence $\{\mu_n\}$ of Borel measures on $[0,1]$ such that $\mu_n(f) \to m(f)$ for all continuous function f on $[0,1]$ but not for all Borel measurable functions f.

(4) Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Let $\{f_n\}$ be an orthonormal set in H such that

$$\sum_{n=1}^{\infty} \|f_n - e_n\| < 1.$$

Show that $\{f_n\}$ is also an orthonormal basis for H.

(5) Let B be a Banach space and H a proper closed subspace of B. Show that for any $\epsilon > 0$, there is an element $x \in B$ such that $\|x\| = 1$ and

$$d(x, H) = \inf_{h \in H} \|x - h\| \geq 1 - \epsilon.$$

Part III. Do four of the following five problems.

(1) Let f be analytic in a neighborhood of the closed unit disc $\overline{D(0;1)}$.
  (a) Suppose that $|f(z)| < 1$ for $|z| = 1$. Show that there exists a unique $z_0 \in D(0;1)$ such that $f(z_0) = z_0$.
  (b) Is this true if $|f(z)| \leq 1$ when $|z| = 1$? Prove that your answer is correct.

(2) Let $t \neq 0$ be a non-zero real number and let $s > 0$ be a positive real number. Use the method of residues to calculate the limit

$$\theta(t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{s-iT}^{s+iT} \frac{e^{tz}}{z} dz.$$

The line integral is along the line segment from $s - iT$ to $s + iT$.

(3) Let $\{f_n\} \subset A(U)$, the space of analytic function on a connected open set $U \subset \mathbb{C}$. Assume that $f_n \to f$ pointwise on U. Show that there exists a dense open set $\Omega \subset U$ so that $f_n \to f$ uniformly on compact subsets of Ω. (Hint: Let

$$A_N = \{z \in U : |f_n(z)| \leq N, \ \forall n = 1, 2, \ldots\}.$$

Use the Baire category theorem to show that some A_N contains a disk D. Let Ω be the union of all disks D such that $f_n \to f$ uniformy on compact subsets of D).

(4) Suppose that $f : D(0;1) \to D(0;1)$ is a holomorphic map. Show that for all $z \in D(0;1)$,

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

(5) Suppose that f and g are entire such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant c so that $f = cg$.