Analysis Prelim, June 2012.

There are 3 parts to this exam: I (measure theory); II (functional analysis); III (complex analysis). Do 3 problems from each part.

General remarks: In each problem, full credit requires proving that your answer is correct. You may quote and use theorems from the course. But if a problem asks you to state or prove a theorem from the text/course, you need to provide the full details.

Part I. Do three of the following five problems.

(1) (a) State the monotone convergence theorem and Fatou's lemma.
(b) Give a proof of Fatou's lemma. You can use the monotone convergence theorem.

(2) Let f be a nonnegative integrable function on a measure space $(X, \mathscr{F}, \mu)$. Show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_E f\, d\mu \leq \epsilon$$

for every measurable set E with $\mu(E) \leq \delta$.

(3) Let $(X, \mathscr{F}, \mu)$ be a measure space of total measure $\mu(X) = 1$. Suppose that f and g are two nonnegative measurable functions such that

$$\lambda\mu\{f \geq \lambda\} \leq \int_X g I_{\{f\geq\lambda\}}\, d\mu$$

for all $\lambda \geq 0$. Show that

$$\int_X f\, d\mu \leq \frac{e}{e-1}\left[1 + \int_X g \ln^+ g\, d\mu\right],$$

where $\ln^+ g = \ln g$ if $g \geq 1$ and $\ln^+ g = 0$ otherwise.

(4) Let $\{\mu_n\}$ and μ be a sequence of Borel measures on $\mathbb{R}$ such that $\mu_n(\mathbb{R}) < \infty$ and $\mu(\mathbb{R}) < \infty$. Suppose that

$$\lim_{n\to\infty} \mu_n(O) = \mu(O)$$

for every open set $O \subset \mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\, d\mu_n = \int_{\mathbb{R}} f\, d\mu$$

for every bounded continuous function on $\mathbb{R}$.

(5) Let $f_n : [0,1] \to \mathbb{R}_+$ be a sequence of nondecreasing functions such that $f_n(0) = 0$ and $\sum_{n=1}^{\infty} f_n(1) < \infty$. Show that the sum can be differentiated term by term, namely,

$$f' = \sum_{n=1}^{\infty} f_n'$$

almost everywhere, where

$$f = \sum_{n=1}^{\infty} f_n.$$

Part II. Do three of the following five problems.

(1) Let S^1 be the unit circle and let

$$\mathcal{P}_N = \{ \sum_{n=-N}^{N} a_n e^{in\theta}, \quad a_n \in \mathbb{C}\} \subset L^2(S^1)$$

be the space of trigonometric polynomials of degree n with complex coefficients. Let $f \in C^1(S^1)$. Show that

$$d_2(f, \mathcal{P}_N) \le \frac{C}{\sqrt{N}}$$

where $d_2(f,g) = ||f - g||_{L^2}$ is the distance in $L^2(S^1)$ and $d_2(f,V) = \inf_{g \in V} d_2(f,g)$ is the distance from f to a closed subspace.

(2) Define the term "separable Banach space". Is $L^\infty([0,1], dx)$ separable? Prove that your answer is correct.

(3) This problem is about several notions of convergence $f_n \to f$ of a sequence $\{f_n\} \subset L^1(\mathbb{R})$.

(a) Norm (or strong) convergence in L^1.
(b) Pointwise convergence almost everywhere: $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere.
(c) Convergence in measure.
(d) Weak convergence: $f_n \to f$ weakly in L^1.
(e) Weak* convergence when $||f_n||_{L^1} \le 1$.

For each of the following sequences, say whether or not it converges in each sense above. If it converges, state what the limit is. Prove that your answers are correct.

The notation $\mathbf{1}_E$ standards for the characeristic (indicator) function of a set E:

- $f_n = \mathbf{1}_{[n,n+1]}$.
- $f_n = n\mathbf{1}_{[-\frac{1}{n},\frac{1}{n}]}$.
- $f_n = \frac{1}{2n}\mathbf{1}_{[-n,n]}$.

- $$f_n = \mathbf{1}_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}$$

  when $2^k \leq n < 2^{k+1}$. Here, $k = 0, 1, 2, \ldots$. (Draw a picture).

(4) Let α, β, and γ be three positive numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Show that

$$\|fgh\|_1 \leq \|f\|_\alpha \|g\|_\beta \|h\|_\gamma.$$

Give a necessary and sufficient condition for the equality to hold.

(5) Let f be a real-valued measurable function on $\mathbb{R}$ such that for every $g \in L^1(\mathbb{R})$ the product $fg \in L^1(\mathbb{R})$. Show that $f \in L^\infty(\mathbb{R})$.

Part III: Do three of the following 5 problems.

(1) Denote by $\mathcal{F}_a$ the class of all holomorphic functions in the strip

$$S_a = \{z \in \mathbb{C} : |\Im z| < a\}$$

and with the decay condition that there exists $A > 0$ so that

$$|f(x+iy)| \leq \frac{A}{1+x^2}, \quad \forall x \in \mathbb{R}, |y| < a.$$

Let $\hat{f}$ be the Fourier transform of f.

Prove the following: If $f \in \mathcal{F}_a$ then $|\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|}$ for all $0 \leq b < a$. Conversely, if $|\hat{f}(\xi)| \leq Be^{-2\pi a|\xi|}$ then f admits a holomorphic continuation to S_b for all $b < a$.

(2) Let $D_r(0)$ be the disc of radius r centered at 0 and let f be holomorphic in a neighborhood of $\overline{D_r}$. Let

$$N(w; r) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f'(\zeta)}{f(\zeta) - w}.$$

- Say in words what $N(w, r)$ equals. What are its possible values?
- Let r be fixed and let w vary. At which w is $N(w, r)$ discontinuous?
- Let w be fixed and let r increase. At which r is $N(w, r)$ discontinuous?
- Prove that your answers are correct.

(3) Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain. Let $a \in \Omega$ and suppose that $f : \Omega \to \Omega$ is an analytic function such that $f(a) = a, |f'(a)| = 1$. Prove that f is 1-1 and onto.

(4) Let $k \geq 1$ be a positive integer. Show that $z^k + kz$ is 1-1 on the unit disc.

(5) Let P, Q be monic polynomials (i.e. $a_d z^d + \cdots a_0$ with $a_d = 1$). Assume that $\deg Q = \deg P + 2$ and consider the rational function $R = \frac{P}{Q}$. What kind of singularities does R have? How many? Calculate

$$\sum_{z \in \mathbb{C}} \operatorname{Res}(R, z).$$