PRELIMINARY EXAM IN ANALYSIS SPRING 2013

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Each part has five problems. Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

Problem I.1. Define the function $g(\xi)$ on $\mathbb{R}$ by

$$g(\xi) = \int_{\mathbb{R}} \frac{e^{ix\xi}}{(1+x^2)^2}\, dx$$

Using the convergence theorems, prove that $g \in C^1(\mathbb{R})$ and show that $|g'(\xi)| \le 1$.

Problem 1.2. This problem has four parts.

(a) State Fatou's lemma.

(b) Show that Fatou's Lemma fails for the sequence of functions on the real line $\mathbb{R}$

$$f_n = -\frac{1}{n}\mathbf{1}_{[n,2n]}.$$

(c) Find a condition for a sequence $\{f_n\}$ of general signed functions under which

$$\limsup_{n\to\infty} \int_X f_n d\mu \le \int_X \limsup_n f_n d\mu.$$

(d) Find a sequence of functions $f_n \ge 0$ on $[0,1]$ which is uniformly integrable and

$$\liminf_{n\to\infty} \int_0^1 f_n dx > \int_0^1 \liminf f_n dx,$$

i.e., for which Fatou's Lemma is a strict inequality.

Problem I.3. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^d, m)$, where m is the Lebesgue measure. Let $f_h(x) = f(x+h)$. Show that

$$\lim_{h\to 0} \|f_h - f\|_p = 0.$$

Problem I.4. Let (X,μ) be a measure space and $1 \le p < \infty$. Suppose that $f : X \times X \to \mathbb{R}_+$ is measurable and nonnegative on $X \times X$. Show that

$$\left\| \int_X f(\cdot,y)\mu(dy) \right\|_p \le \int_X \|f(\cdot,y)\|_p\, \mu(dy).$$

Problem I.5. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Define the density of E at x by

$$D_E(x) = \lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

if the limit exists.

(a) Show that $D_E(x) = 1$ for Lebesgue almost every point of E.

(b) Show that $D_{E^c}(x) = 0$ for Lebesgue almost every point of $E^c = \mathbb{R} \backslash E$.

(c) Find an example of E and x for which $D_E(x) = 1/2$.

Part II. Functional Analysis

Do **three** of the following five problems.

Problem II.1. This problem has three parts.

(a) Define "$f_n \to f$ weakly in $L^p(X, \mu)$".

(b) Let $f_n(x) = n^{1/p} I_{[0,1]}(nx)$. Show that $f_n \to 0$ weakly if $p > 1$ but not if $p = 1$;.

(c) Show that if $f_n \to f$, a.e. and $\|f_n\|_p \leq M$ for some fixed constant M and all n, then $f_n \to f$ weakly in $L^p(X, \mu)$. Show that this may fail if $p = 1$. (You may want to break up X into sets, using Egorov's theorem and also using that if $g \in L^q$ then $|g|^q d\mu$ is absolutely continuous with respect to μ).

Problem II.2. This problem has three parts.

(a) State the Open Mapping Theorem and the Closed Graph Theorem.

(b) Show that the following properties for a bounded linear transformation $T : X \to Y$ of Banach spaces are equivalent:

(1) T is an open map.

(2) There exists $C > 0$ such that for all $y \in Y$, there exists a solution $x \in X$ of $Tx = y$ satisfying $||x||_X \leq C||y||_Y$.

(c) Suppose that $T : X \to Y$ is a surjective bounded linear transformation of Banach spaces. Show that the transpose map $T^* : Y^* \to X^*$ is bounded from below, i.e. $||T^*\lambda||_{X^*} \geq c||\lambda||_{Y^*}$ for some $c > 0$. (Here, X^* is the dual space of X, etc.).

Problem II.3. Let H be a Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis. Let $K \subset H$ be a subset. For $f \in K$ write $f \sim \sum_{n=1}^{\infty} a_n e_n$ for the Fourier series with respect to $\{e_n\}$. Show that K is compact if and only if it is closed, bounded and its elements have equi-small tails, i.e. for all $\epsilon > 0$ there exists p such that $\sum_{n \geq p} |a_n|^2 < \epsilon$ for all elements of K.

Problem II.4. This problem has two parts. Let $K(x, y) = |x - y|^{-1/2}$ and define

$$Tf(x) = \int_{\mathbb{R}} K(x, y) f(y) dy \quad \text{for } f \in C[0, 1].$$

(a) Prove that T extends to a bounded operator on $L^2[0, 1]$.

(b) Define the term "Hilbert-Schmidt operator". Is T a Hilbert-Schmidt operator?

Problem II.5. This problem has three parts. Let μ, ν be (positive) finite measures on a measurable space $(X, \mathcal{M})$.

(a) Define the term " $\nu \ll \mu$ (ν is absolutely continuous with respect to μ).

(b) Prove that there exists $f \in L^1(X, \mu)$ such that $d\nu = f\,d\mu$ (the Radon-Nikodym theorem). You may use the following outline: Show that $\ell_\nu(\phi) = \int_X \phi\,d\nu$ is a bounded linear functional on $L^2(X, \mu + \nu)$. Find f using the Riesz representation theorem for Hilbert spaces.

Part III. Complex Analysis

Do **three** of the following five problems.

Problem III.1. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the punctured complex plane. Find all conformal equivalences of the punctured complex plane $\mathbb{C}^*$ to itself.

Problem III.2. This problem has two parts.

(a) Let $f(x) = e^{-x^2}$ be the Gaussian function. Compute its Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{i\xi x}\,dx.$$

You may use the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

(b) Use the calculus of residues to compute the complex line integral

$$\oint_C \frac{z}{(z^2-1)(z^2+1)}\,dz,$$

where $C = \left\{(x, y) \in \mathbb{C} : x^2 + y^2 - 2x - 2 = 0\right\}$ in the counterclockwise direction.

Problem III.3. This problem has two parts. Let u be a harmonic function on $\mathbb{R}^2$.

(a) Show that there is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that the real part of f is equal to u.

(b) Show that if there is a constant C such that $u(x, y) \geq C$ for all $(x, y) \in \mathbb{R}^2$, then u must be a constant.

Problem III.4. Let $U \subset \mathbb{C}$ be a connected domain in the complex plane and $\{f_n\}$ a sequence of holomorphic functions on U such that $f_n(z) \to f(z)$ for every $z \in U$ and uniformly on every compact subset of U. Suppose that $f_n(z) \neq 0$ for all n and all $z \in U$. Show that either $f(z) = 0$ for all $z \in U$ or f never vanishes on U.

Problem III.5. Let

$$f_N(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^N}{N!}.$$

Let z_N be the zero of f_N closest to the origin. Show that there is a positive constant C such that $|z_N| \geq CN$ for all N. You may use Stirling's formula

$$\sqrt{2\pi N}\left(\frac{N}{e}\right)^N < N! < \sqrt{2\pi N}\left(\frac{N}{e}\right)^N e^{1/12N}.$$