Solve two out of three problems in each of the following three sections A, B and C. If you attempt to solve more than two problems in a section please indicate which two you want graded.

A.

1. State and prove Egorov's theorem.
2. For $0 < p < 1$ define

$$L^p(R^n) = \{f : R^n \to R,\ measurable,\ \int_{R^n} |f(x)|^p dx < \infty\}$$

Show that $L^p(R^n)$ is a complete metric vector space with respect to the distance

$$d(f,g) = \int_{R^n} |f(x) - g(x)|^p dx$$

3. State and prove the Hahn-Banach theorem.

B.

1. True or false: There is a measure in R with respect to which all functions $f : R \to R$ are measurable.
2. True or false: If $f_n, f \in L^p(R)$, $f_n \to f$ a.e. in R then $\lim_{n\to\infty} \|f_n\|_{L^p} = \|f\|_{L^p}$.
3. True or false: If $f_n \rightharpoonup f$ (weak convergence) in $L^2(R^n)$ and $\|f_n\|_{L^2} \to \|f\|_{L^2}$ then $f_n \to f$ in $L^2(R^n)$ (strong convergence).

C.

1. Let $P(z)$ be a polynomial. Prove that all zeros of its derivative $P'(z)$ lie in the smallest convex polygon that contains all the zeros of the polynomial $P(z)$.
2. Evaluate the integral

$$\int_0^\infty \frac{x^{1-\alpha}}{1+x^2} dx, \qquad 0 < \alpha < 2.$$

3. Let $f(z)$ be a function that is analytic in the unit disk $|z| < 1$. Suppose that $|f(z)| \le 1$ in the unit disk. Prove that if $f(z)$ has at least two fixed points z_1 and z_2 (that is $f(z_j) = z_j, j = 1, 2$), then $f(z) = z$ for all z in the unit disk.